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ORIGINAL ARTICLE

Fekete-Szegö inequalities for *p*-valent starlike and convex functions of complex order

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Abstract In this paper, we obtain Fekete-Szegö inequalities for certain class of analytic p-valent functions $f(z)$ for which $1+\frac{1}{b}\left[\frac{1}{p}\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-1\right]\prec\varphi(z)(b\in\mathbb{C}^*=\mathbb{C}\setminus\{0\})$. Sharp bounds for the Fekete-Szegö functional $|a_{p+2}-\mu a_{p+1}^2|$ are obtained.

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1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k\ (p\in\mathbb{N}=\{1,2,\ldots\}),\tag{1.1}$$

which are analytic and p-valent in the open unit disk $U=\{z:z\in\mathbb{C}\text{ and }|z|<1\}$. Let $g(z)\in\mathcal{A}(p)$, be given by

$$g(z)=z^p+\sum_{k=p+1}^{\infty}g_kz^k.\tag{1.2}$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z^p+\sum_{k=p+1}^{\infty}a_kg_kz^k=(g*f)(z).\tag{1.3}$$

A function $f(z)\in\mathcal{A}(p)$ is said to be p-valent starlike of order α, denoted by $S_p^*(\alpha)$, if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\ (0\leqslant\alpha<p;z\in U).\tag{1.4}$$

A function $f(z)\in\mathcal{A}(p)$ is said to be p-valent convex of order α, denoted by $C_p(\alpha)$, if and only if

$$\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\ (0\leqslant\alpha<p;z\in U).\tag{1.5}$$

The classes $S_p^*(\alpha)$ and $C_p(\alpha)$ were defined by Owa [1]. From (1.4) and (1.5), it follows that

$$f(z)\in C_p(\alpha)\Longleftrightarrow\frac{zf'(z)}{p}\in S_p^*(\alpha).\tag{1.6}$$

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For two functions f and g, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U and write $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)) (z \in U)$. Furthermore, if the function g is univalent in U, then we have the following equivalence (see [2]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\varphi(z)$ be an analytic function with positive real part on U satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $S^*_{b,p}(\varphi)$ be the class of functions $f(z) \in \mathcal{A}(p)$ for which

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) \prec \varphi(z) \ (z \in U), \tag{1.7}$$

and $C_{b,p}(\varphi)$ be the class of functions $f(z) \in \mathcal{A}(p)$ for which

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \ (z \in U). \tag{1.8}$$

The classes $S^*_{b,p}(\varphi)$ and $C_{b,p}(\varphi)$ were introduced and studied by Ali et al. [3]. We note that $S^*_{1,1}(\varphi) = S^*(\varphi)$ and $C_{1,1}(\varphi) = C(\varphi)$, the classes $S^*(\varphi)$ and $C(\varphi)$ were introduced and studied by Ma and Minda [4]. The classes $S^*(\alpha)$ and $C(\alpha)$ are the special cases of $S^*(\varphi)$ and $C(\varphi)$, respectively, when $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \leqslant \alpha < 1)$.

For $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $0 \leqslant \lambda \leqslant 1$ and $p \in \mathbb{N}$, we let $S_{\lambda,b,p}(g, \varphi)$ be the subclass of $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (1.1), the functions $g(z)$ of the form (1.2) with $g_k > 0$ and satisfying the analytic criterion:

$$1 + \frac{1}{b}\left[\frac{1}{p}\frac{z(f*g)'(z) + \lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)} - 1\right] \prec \varphi(z). \tag{1.9}$$

We note that for suitable choices of $g(z)$, λ, b, p and $\varphi(z)$, we obtain the following subclasses:

(1) $S_{0,b,p}(g, \varphi) = S^*_{b,p,g}(\varphi)$ (see Ali et al. [3]);
(2) $S_{0,b,p}\left(\frac{z^p}{1-z}, \varphi(z)\right) = S^*_{b,p}(\varphi)$ and $S_{1,b,p}\left(\frac{z^p}{1-z}, \varphi(z)\right) = C_{b,p}(\varphi)$ (see Ali et al. [3]);
(3) $S_{0,1,p}(g, \varphi) = S^*_{p,g}(\varphi)$ and $S_{0,1,p}\left(\frac{z^p}{1-z}, \varphi(z)\right) = S^*_p(\varphi)$ (see Ali et al. [3]);
(4) $S_{0,b,1}\left(\frac{z}{1-z}, \varphi(z)\right) = S^*_b(\varphi)$ and $S_{1,b,1}\left(\frac{z}{1-z}, \varphi(z)\right) = C_b(\varphi)$ (see Ravichandran et al. [5]);
(5) $S_{0,1,1}\left(\frac{z}{1-z}, \varphi(z)\right) = S^*(\varphi)$ (see Ma and Minda [4] and Shanmugam and Sivasubramanian [6, with $\alpha = 0$]);
(6) $S_{1,1,1}\left(\frac{z}{1-z}, \varphi(z)\right) = C(\varphi)$ (see Ma and Minda [4] and Shanmugam and Sivasubramanian [6, with $\alpha = 1$]);
(7) $S_{0,\left(1-\frac{\gamma}{p}\right)e^{-i\alpha}\cos\alpha,p}\left(\frac{z^p}{1-z}, \frac{1-z}{1+z}\right) = S^\alpha(p,\gamma) (|\alpha| < \frac{\pi}{2};\ 0 \leqslant \gamma < p)$ (see Patil and Thakare [7]).

Also, we note that:

(1)
$$S_{\lambda,\left(1-\frac{\gamma}{p}\right)e^{-i\alpha}\cos\alpha,p}\left(\frac{z^p}{1-z}, \varphi\right) = S^\alpha_{\lambda,p,\gamma}(\varphi) = \left\{f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left[\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right] - \gamma\cos\alpha - ip\sin\alpha}{(p-\gamma)\cos\alpha} \prec \varphi(z) \left(|\alpha| < \frac{\pi}{2};\ 0 \leqslant \gamma < p; 0 \leqslant \lambda \leqslant 1\right)\right\};$$

(2)
$$S_{0,\left(1-\frac{\gamma}{p}\right)e^{-i\alpha}\cos\alpha,p}\left(\frac{z^p}{1-z}, \varphi\right) = S^\alpha_{p,\gamma}(\varphi) = \left\{f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left(\frac{zf'(z)}{f(z)}\right) - \gamma\cos\alpha - ip\sin\alpha}{(p-\gamma)\cos\alpha} \prec \varphi(z)\left(|\alpha| < \frac{\pi}{2}; 0 \leqslant \gamma < p\right)\right\};$$

(3)
$$S_{1,\left(1-\frac{\gamma}{p}\right)e^{-i\alpha}\cos\alpha,p}\left(\frac{z^p}{1-z}, \varphi\right) = C^\alpha_{p,\gamma}(\varphi) = \left\{f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) - \gamma\cos\alpha - ip\sin\alpha}{(p-\gamma)\cos\alpha} \prec \varphi(z) \ \left(|\alpha| < \frac{\pi}{2}; 0 \leqslant \gamma < p\right)\right\};$$

(4)
$$S_{\lambda,b,p}\left(z^p + \sum_{k=p+1}^{\infty}\Gamma_{k,p}(\alpha_1)z^k, \varphi\right) = S_{\lambda,p,b}(\alpha_1;\varphi) = \left\{f \in \mathcal{A}(p) : 1 + \frac{1}{b}\left[\frac{1}{p}\frac{z(H_{p,\ell,m}(\alpha_1)f(z))' + \lambda z^2(H_{p,\ell,m}(\alpha_1)f(z))''}{(1-\lambda)(H_{p,\ell,m}(\alpha_1)f(z)) + \lambda z(H_{p,\ell,m}(\alpha_1)f(z))'} - 1\right] \prec [\varphi(z) \ (\ell \leqslant m+1; \ell, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})\right\}$$

where the operator

$$H_{p,\ell,m}(\alpha_1)(z) = z^p + \sum_{k=p+1}^{\infty}\Gamma_{k,p}(\alpha_1)z^k,$$

$$\Gamma_{k,p}(\alpha_1) = \frac{(\alpha_1)_{k-p}\cdots(\alpha_\ell)_{k-p}}{(\beta_1)_{k-p}\cdots(\beta_m)_{k-p}}\frac{1}{(k-p)!}, \tag{1.10}$$

$\alpha_1, \ldots, \alpha_\ell$ and $\beta_1, \ldots, \beta_m$ are real parameters, $\beta_j \neq 0, -1, -2, \ldots; j = 1, \ldots, m$, was introduced and studied by Dziok and Srivastava [8];

(5)
$$S_{\lambda,b,p}\left(z^p + \sum_{k=p+1}^{\infty}\left[\frac{p+\ell+\gamma(k-p)}{p+\ell}\right]^m z^k, \varphi(z)\right) = S_{\lambda,p,b}(m,\gamma,\ell;\varphi) = \left\{f \in \mathcal{A}(p) : 1 + \frac{1}{b}\left[\frac{1}{p}\frac{z\left(I^m_p(\gamma,\ell)f(z)\right)' + \lambda z^2\left(I^m_p(\gamma,\ell)f(z)\right)''}{(1-\lambda)\left(I^m_p(\gamma,\ell)f(z)\right) + \lambda z\left(I^m_p(\gamma,\ell)f(z)\right)'} - 1\right] \prec \varphi(z)(\gamma \geqslant 0; \ell \geqslant 0; m \in \mathbb{Z}; p \in \mathbb{N})\right\};$$

where the operator

$$I^m_p(\gamma,\ell)(z) = z^p + \sum_{k=p+1}^{\infty}\left[\frac{p+\ell+\gamma(k-p)}{p+\ell}\right]^m z^k, \tag{1.11}$$

was introduced and studied by Prajapat [9], (see also, Catas [10] and El-Ashwah and Aouf [11] with $m \in \mathbb{N}_0$).

In this paper, we obtain the Fekete-Szegö inequalities for functions in the class $S_{\lambda,b,p}(g, \varphi)$.