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ORIGINAL ARTICLE

On some first-order differential subordination

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Abstract Let $\mathcal{A}$ denote the class of functions f that are analytic in the unit disc $\mathbb{D}$ and normalized by $f(0) = f'(0) - 1 = 0$. In this paper, we investigate the class of functions such that $\Re\mathfrak{e}\{f'(z) + zf''(z) - \beta\} > \alpha$ in $\mathbb{D}$. We determine conditions for α and β under which the function f is univalent, close-to-convex, and convex. To obtain this, we first estimate $|\mathrm{Arg}\{f'(z)\}|$ which improves the earlier results.

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1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and denote by $\mathcal{A}$ the class of analytic functions in $\mathbb{D}$ and usually normalized, i.e., $\mathcal{A} = \{f \in \mathcal{H} : f(0) = 0, f'(0) = 1\}$. We say that the $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ in the unit disc $\mathbb{D}$, written $f \prec g$ if and only if there exists an analytic function $w \in \mathcal{H}$ such that $|w(z)| \leqslant |z|$ and $f(z) = g[w(z)]$ for $z \in \mathbb{D}$. Therefore, $f \prec g$ in $\mathbb{D}$ implies $f(\mathbb{D}) \subseteq g(\mathbb{D})$. In particular, if g is univalent in $\mathbb{D}$ then the Subordination Principle says that $f \prec g$ if and only if $f(0) = g(0)$ and $f(|z| < r) \subseteq g(|z| < r)$, for all $r \in (0,1)$.

The class $\mathcal{S}^*_\alpha$ of starlike functions of order $\alpha < 1$ may be defined as

$$\mathcal{S}^*_\alpha := \{f \in \mathcal{A} : \Re\mathfrak{e}\frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D}\}.$$

The class $\mathcal{S}^*_\alpha$ and the class $\mathcal{K}_\alpha$ of convex functions of order $\alpha < 1$

$$\mathcal{K}_\alpha := \{f \in \mathcal{A} : \Re\mathfrak{e}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{D}\}$$
$$= \{f \in \mathbb{D} : zf' \in \mathcal{S}^*_\alpha\}$$

were introduced by Robertson in [11]. If $\alpha \in [0;1)$, then a function in either of these sets is univalent. In particular, we denote $\mathcal{S}^*_0 = \mathcal{S}^*, \mathcal{K}_0 = \mathcal{K}$, the classes of starlike and convex functions, respectively. Recall that $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}_\alpha(\beta)$,

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[3], of close-to-convex functions of order β and type α, $0 \leqslant \beta < 1$, if and only if there exist $g \in \mathcal{K}_\alpha, \varphi \in \mathbb{R}$, such that

$$\Re\left\{e^{i\varphi}\frac{f'(z)}{g'(z)}\right\} > \beta, \quad z \in \mathbb{D}. \tag{1.1}$$

Functions defined by (1.1) with $\varphi = 0$ were considered earlier by Ozaki [10], see also Umezawa [12,13]. Moreover, Lewandowski [5,6] defined the class of functions $f \in \mathcal{A}$ for which the complement of $f(\mathbb{D})$ with respect to the complex plane is a linearly accessible domain in the large sense. The Lewandowski's class is identical with the Kaplan's class $\mathcal{C}_0(0)$.

2. Main result

Theorem 2.1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc $\mathbb{D}$. If*

$$f'(z) \neq 0,\ f'(z) + zf''(z) \neq 0, \quad z \in \mathbb{D} \tag{2.1}$$

and

$$\Re\{f'(z) + zf''(z)\} > \beta, \quad z \in \mathbb{D}, \tag{2.2}$$

then

$$|\mathrm{Arg}\{f'(z)\}| \leqslant \begin{cases} \frac{\pi(1-\beta)}{2(1-2\beta)}\log\{2(1-\beta)\} & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ \frac{\pi}{2} - 1 & \beta = 1/2. \end{cases} \tag{2.3}$$

Moreover, f is close-to-convex in $\mathbb{D}$ whenever

$\beta > \beta_0$,

where $-1.47 < \beta_0 < -1.46$ is the positive solution of the equation

$$\log\{2(1-\beta)\} = \frac{1-2\beta}{1-\beta}. \tag{2.4}$$

Proof. Note that the assumptions (2.1) are necessary for $\beta < 0$ only. If $\beta \in [0, 1)$, then from (2.2) we have even more $\Re\{f'(z) + zf''(z)\} > 0$. Moreover, from (2.2) we have also that $\Re\{(zf'(z))'\} > 0$ so zf' is univalent in $\mathbb{D}$ and $f'(z) \neq 0$.

From the hypothesis (2.2), we have

$$\frac{f'(z) + f''(z) - \beta}{1-\beta} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}$$

and so, it follows that

$$f'(z) + zf''(z) \prec (1-\beta)\frac{1+z}{1-z} + \beta, \quad z \in \mathbb{D}. \tag{2.5}$$

From (2.5), we have

$$\begin{aligned} &|\mathrm{Arg}\{f'(\rho e^{i\theta}) + \rho e^{i\theta} f''(\rho e^{i\theta})\}| \\ &\quad \leqslant \sin^{-1}\left\{\frac{2(1-\beta)\rho}{1+(1-2\beta)\rho^2}\right\} \quad \text{for all} \quad \rho \in [0, 1), \\ &\quad \theta \in (-\pi, \pi]. \end{aligned} \tag{2.6}$$

On the other hand, it follows that

$$\begin{aligned} f'(z) &= \frac{zf'(z)}{z} \\ &= \frac{1}{z}\int_0^z (tf'(t))' \mathrm{d}t \\ &= \frac{1}{z}\int_0^z (f'(t) + tf''(t))\mathrm{d}t \\ &= \frac{1}{re^{i\theta}}\int_0^r (f'(\rho e^{i\theta}) + \rho e^{i\theta} f''(\rho e^{i\theta}))e^{i\theta}\mathrm{d}\rho \\ &= \frac{1}{r}\int_0^r (f'(\rho e^{i\theta}) + \rho e^{i\theta} f''(\rho e^{i\theta}))\mathrm{d}\rho, \end{aligned} \tag{2.7}$$

where $z = \rho e^{i\theta}$, $\rho \in [0, 1)$, $\theta \in (-\pi, \pi]$. It is known that

$$\sin^{-1} x \leqslant \frac{\pi}{2} x \quad \text{for} \quad x \in [0, 1]. \tag{2.8}$$

Then, applying the same idea of [9, pp. 1292–1293], Theorem 2.2, applying also (2.5), (2.7) and (2.8), we have

$$\begin{aligned} |\mathrm{Arg}\{f'(z)\}| &= \left|\mathrm{Arg}\left\{\frac{1}{r}\int_0^r (f'(\rho e^{i\theta}) + \rho e^{i\theta} f''(\rho e^{i\theta}))\mathrm{d}\rho,\right\}\right| \\ &\leqslant \int_0^r |\mathrm{Arg}\{f'(\rho e^{i\theta}) + \rho e^{i\theta} f''(\rho e^{i\theta})\}|\mathrm{d}\rho, \\ &\leqslant \int_0^r \sin^{-1}\{\frac{2(1-\beta)\rho}{1+(1-2\beta)\rho^2}\}\mathrm{d}\rho \\ &\leqslant \begin{cases} \frac{\pi}{2}\int_0^r \left\{\frac{2(1-\beta)\rho}{1+(1-2\beta)\rho^2}\right\}\mathrm{d}\rho & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ \int_0^r \sin^{-1}\rho \mathrm{d}\rho & \beta = 1/2, \end{cases} \\ &= \begin{cases} \frac{\pi(1-\beta)}{2(1-2\beta)}\int_0^r \left\{\frac{2(1-2\beta)\rho}{1+(1-2\beta)\rho^2}\right\}\mathrm{d}\rho & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ \int_0^r \sin^{-1}\rho \mathrm{d}\rho & \beta = 1/2, \end{cases} \\ &= \begin{cases} \frac{\pi(1-\beta)}{2(1-2\beta)}\{\log\{1+(1-2\beta)\rho^2\}\}\Big|_{\rho=0}^{\rho=r} & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ \left\{\rho\sin^{-1}\rho + \sqrt{1-\rho^2}\right\}\Big|_{\rho=0}^{\rho=r} & \beta = 1/2, \end{cases} \\ &= \begin{cases} \frac{\pi(1-\beta)}{2(1-2\beta)}\log\{1+(1-2\beta)r^2\} & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ r\sin^{-1} r + \sqrt{1-r^2} - 1 & \beta = 1/2. \end{cases} \end{aligned}$$

Letting $r \to 1^-$ we obtain

$$|\mathrm{Arg}\{f'(z)\}| \leqslant \begin{cases} \frac{\pi(1-\beta)}{2(1-2\beta)}\log\{2(1-\beta)\} & \beta \in (-\infty, 1/2) \cup (1/2, 1), \\ \frac{\pi}{2} - 1 & \beta = 1/2. \end{cases}$$

It is easy to see that there exists β_0, $-1.47 < \beta_0 < -1.46$, such that

$$\frac{\pi(1-\beta_0)}{2(1-2\beta_0)}\log\{2(1-\beta_0)\} = \frac{\pi}{2}$$

and so for $\beta > \beta_0$, we have

$\Re\{f'(z)\} > 0, \ z \in \mathbb{D}$.

This means that f is a close-to-convex function with respect to $g(z) = z$, see (1.1). It completes the proof. □

Recall here the well known theorem due to Hallenbeck and Ruscheweyh [2].

Theorem A (see [2]). *Let the function h be analytic and convex univalent in $|z| < 1$ with $h(0) = a$. Let also $p(z) = a + b_n z^n + b_{n+1} z^{n+1} + \cdots$ be analytic in $\mathbb{D}$ If*

$$p(z) + \frac{zp'(z)}{c} \prec h(z), \quad z \in \mathbb{D}$$

for $\Re\{c\} \geqslant 0, c \neq 0$, then

$$p(z) \prec q_n(z) \prec h(z), \quad z \in \mathbb{D},$$

where $q_n(z) = \frac{c}{nz^{c/n}}\int_0^z t^{c/n-1}h(t)\mathrm{d}t$. Moreover, the function $q_n(z)$ is convex univalent and is the best dominant of $p \prec q_n$ in the sense that if $p \prec q$, then $q_n \prec q$.

The condition (2.2) becomes

$$f'(z) + zf''(z) \prec h_\beta(z) = (1-\beta)\frac{1-z}{1+z} + \beta$$

where h_β is convex univalent and maps the unit disc onto the half-plane $\Re\{w\} > \beta$. Using the above theorem with $n = 1$, $c = 1$, we immediately get