Journal of the Egyptian Mathematical Society (2014) **22**, 209–213

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Quintic B-spline for the numerical solution of the good Boussinesq equation

Shahid S. Siddiqi *, **Saima Arshed**

Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan

Received 15 March 2013; revised 21 May 2013; accepted 29 June 2013
Available online 22 August 2013

KEYWORDS

Quintic B-spline;
Collocation method;
Stability analysis;
Soliton;
Finite difference;
Boussinesq equation

Abstract A numerical method is developed to solve the nonlinear Boussinesq equation using the quintic B-spline collocation method. Applying the Von Neumann stability analysis, the proposed method is shown to be unconditionally stable. An example has been considered to illustrate the efficiency of the method developed.

MATHEMATICS SUBJECT CLASSIFICATION (2010): 35-XX; 65-XX; 34B15; 34B05

1. Introduction

The nonlinear Boussinesq equation, which belongs to the KdV family of equations, describes shallow water waves propagating in both directions, is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 g(u)}{\partial x^2} + q\frac{\partial^4 u}{\partial x^4}, \quad x \in [a,b], \ t > t_0. \tag{1.1}$$

where $u = u(x,t)$, $g(u) = u(1+u)$ and $|q| = 1$ is a real parameter. Taking $q = -1$ gives the good Boussinesq or well-posed equation (GB), while taking $q = 1$ gives the bad Boussinesq or ill-posed equation (BB).

The initial displacement associated with Eq. (1.1) is given by

$$u(x,t_0) = f(x)$$

with initial velocity,

$$u_t(x,t_0) = f_1(x) \tag{1.2}$$

Wazwaz [1] developed modified decomposition method for construction of soliton solutions and periodic solutions of the Boussinesq equation. Bratsos [2] presented the method of lines for the numerical solution of the Boussinesq equation. Bratsos [3] derived a parametric scheme for the numerical solution of the Boussinesq equation. Choo [4] proposed pseudo-spectral method for the damped Boussinesq equation. Daripa and Hua [5] have used filtering and regularization techniques, for the numerical study of an ill-posed Boussinesq equation arising in water waves and nonlinear lattices. Ismail and Bratsos [6] presented a predictor-corrector scheme for the numerical solution of the Boussinesq equation. Tzirtzilakis et al. [7] proposed spectral methods for the numerical solution of the Boussinesq equation. Al-Khaled and Nusier [8] have used Adomians decomposition method and the Galerkin interpolation methods based on Sinc functions to derive the numerical solution of the Boussinesq equation. Mohyud-Din et al. [9] developed the numerical solution of two-dimensional Boussinesq equation using Adomian Decomposition and He's Homotopy Perturbation Method.

* Corresponding author. Tel.: +92 4299231241.
E-mail addresses: shahidsiddiqiprof@yahoo.co.uk (S.S. Siddiqi), saimaarshed10@gmail.com (S. Arshed).

1110-256X © 2013 Production and hosting by Elsevier B.V. on behalf of Egyptian Mathematical Society. Open access under CC BY-NC-ND license.
http://dx.doi.org/10.1016/j.joems.2013.06.015

The paper is organized into six sections. A finite difference approximation technique is used to discretize the Eq. (1.1) in time derivatives is discussed in Section 2. Quintic B-spline collocation method to solve the Boussinesq equation is discussed in Section 3. Section 4 presents the way to obtain the initial state which is required to start our scheme. Stability analysis of the proposed method is discussed in Section 5. Numerical results are presented in Section 6.

2. Temporal discretization

Consider the following Boussinesq equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u^2}{\partial x^2} + q\frac{\partial^4 u}{\partial x^4} \tag{2.3}$$

with boundary conditions

$$\begin{aligned} &u(a,t) = 0,\\ &u(b,t) = 0,\\ &u_{xx}(a,t) = 0,\\ &u_{xx}(b,t) = 0 \end{aligned} \tag{2.4}$$

The exact solution of the problem can be written as [10]

$$u(x,t) = q\left[sech^2\left[\sqrt{\frac{A}{6}}(x - ct + x^0)\right] + \left(b - \frac{q}{2}\right)\right]$$

where, A is the amplitude of the pulse, b is an arbitrary parameter, x^0 is the initial position of the pulse, and $c = \pm\left[2q(b + \frac{4}{3})\right]^{1/2}$ is the velocity.

Consider a uniform mesh Δ with the grid points λ_{ij} to discretize the region $\Omega = [a,b] \times [0,T]$. Each λ_{ij} is the vertices of the grid point (x_i, t_j) where $x_i = a + ih$, $i = 0, 1, 2, \ldots, N$ and $t_j = jk$, $j = 0, 1, 2, \ldots, M$, $Mk = T$. The quantities h and k are the mesh size in the space and time directions, respectively.

Approximate the time derivative by usual finite difference formula as

$$\frac{\partial^2 u^n}{\partial t^2} = \frac{u^{n+1} - 2u^n + u^{n-1}}{k^2} + O(k^2) \tag{2.5}$$

Substituting the above approximation into equation Eq. (2.3) and discretizing in time variable, the equation becomes

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{k^2} = \frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 (u^2)^n}{\partial x^2} + q\frac{\partial^4 u^n}{\partial x^4} \tag{2.6}$$

Apply θ-weighted scheme to space derivatives to Eq. (2.6), where, $(0 \leqslant \theta \leqslant 1)$, it can be written as

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{k^2} = \theta\left(\frac{\partial^2 u^{n+1}}{\partial x^2} + \frac{\partial^2 (u^2)^{n+1}}{\partial x^2} + q\frac{\partial^4 u^{n+1}}{\partial x^4}\right) + (1 - \theta)\left(\frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 (u^2)^n}{\partial x^2} + q\frac{\partial^4 u^n}{\partial x^4}\right)$$

where the superscripts $n-1$, n, $n+1$ denote the adjacent time levels. Taking θ to be $\frac{1}{2}$, the above equation becomes

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{k^2} = \frac{(u_{xx}^{n+1} + u_{xx}^n)}{2} + q\frac{(u_{xxxx}^{n+1} + u_{xxxx}^n)}{2} + \frac{(u_{xx}^2)^{n+1} + (u_{xx}^2)^n}{2} \tag{2.7}$$

The nonlinear term in Eq. (2.7) can be linearized using Taylor expansion as

$$(u_{xx}^2)^{n+1} = 2u_{xx}^n u_{xx}^{n+1} - (u_{xx}^2)^n \tag{2.8}$$

Substituting Eq. (2.8) into Eq. (2.7), Eq. (2.7) leads to

$$2u^{n+1} - k^2 u_{xx}^{n+1} - k^2 q u_{xxxx}^{n+1} - 2k^2 u_{xx}^n u_{xx}^{n+1} = 4u^n - 2u^{n-1} + k^2 u_{xx}^n + k^2 q u_{xxxx}^n \tag{2.9}$$

The space derivatives are approximated by quintic B-spline, which is presented in the next section.

3. Quintic B-spline collocation method

The interval $[a,b]$ of domain has been subdivided as

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

To provide the support for the quintic B-spline near the end boundaries, ten additional knots have been introduced as

$$x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0$$

and

$$x_N < x_{N+1} < x_{N+2} < x_{N+3} < x_{N+4} < x_{N+5}.$$

The basis functions $B_j(x)$, $j = -2, \ldots, N+2$ of quintic B-spline are defined as

$$B_j(x) = \frac{1}{h^5}\begin{cases} (x - x_j + 3h)^5, & x \in [x_{j-3}, x_{j-2}],\\ (x - x_j + 3h)^5 - 6(x - x_j + 2h)^5, & x \in [x_{j-2}, x_{j-1}],\\ (x - x_j + 3h)^5 - 6(x - x_j + 2h)^5 + 15(x - x_j + h)^5, & x \in [x_{j-1}, x_j],\\ (-x + x_j + 3h)^5 - 6(-x + x_j + 2h)^5 + 15(-x + x_j + h)^5, & x \in [x_j, x_{j+1}],\\ (-x + x_j + 3h)^5 - 6(-x + x_j + 2h)^5, & x \in [x_{j+1}, x_{j+2}],\\ (-x + x_j + 3h)^5, & x \in [x_{j+2}, x_{j+3}],\\ 0 & otherwise \end{cases}$$

The values of successive derivatives $B_j^{(r)}(x), j = -2, \ldots, N+2$; $r = 0, 1, 2, 3, 4$ at nodes are listed in Table 1. For solving Eq. (2.9) using the collocation method with quintic B-spline, an approximate solution $U(x,t)$ to the exact solution of the problem is to be found. Let $U(x,t)$ can be written in the following form

$$U(x,t) = \sum_{j=-2}^{N+2} c_j(t) B_j(x) \tag{3.10}$$

where c_j are unknown real coefficients and $B_j(x)$ are quintic B-spline functions.

Substituting Eq. (3.10) into Eq. (2.9) yields the following equation

$$2\sum_{j=-2}^{N+2} c_j^{n+1}(t) B_j(x_i) - k^2 \sum_{j=-2}^{N+2} c_j^{n+1}(t) B_j''(x_i) - k^2 q \sum_{j=-2}^{N+2} c_j^{n+1}(t) B_j^{(4)}(x_i) - 2k^2 \sum_{j=-2}^{N+2} c_j^n(t) B_j''(x_i) \sum_{j=-2}^{N+2} c_j^{n+1}(t) B_j''(x_i)$$
$$= 4\sum_{j=-2}^{N+2} c_j^n(t) B_j(x_i) - 2\sum_{j=-2}^{N+2} c_j^{n-1}(t) B_j(x_i) + k^2 \sum_{j=-2}^{N+2} c_j^n(t) B_j''(x_i) + k^2 q \sum_{j=-2}^{N+2} c_j^n(t) B_j^{(4)}(x_i)$$

Simplifying, the above relation leads to the following system of $(N+1)$ equations in $(N+5)$ unknowns $\left(c_{-2}^{n+1}, c_{-1}^{n+1}, c_0^{n+1}, c_1^{n+1}, \ldots, c_N^{n+1}, c_{N+1}^{n+1}, c_{N+2}^{n+1}\right)$