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ORIGINAL ARTICLE

Solutions of 2nd-order linear differential equations subject to Dirichlet boundary conditions in a Bernstein polynomial basis

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Abstract An algorithm for approximating solutions to 2nd-order linear differential equations with polynomial coefficients in **B**-polynomials (Bernstein polynomial basis) subject to Dirichlet conditions is introduced. The algorithm expands the desired solution in terms of **B**-polynomials over a closed interval [0, 1] and then makes use of the orthonormal relation of **B**-polynomials with its dual basis to determine the expansion coefficients to construct a solution. Matrix formulation is used throughout the entire procedure. However, accuracy and efficiency are dependent on the size of the set of **B**-polynomials, and the procedure is much simpler compared to orthogonal polynomials for solving differential equations. The current procedure is implemented to solve five linear equations and one first-order nonlinear equation, and excellent agreement is found between the exact and approximate solutions. In addition, the algorithm improves the accuracy and efficiency of the traditional methods for solving differential equations that rely on much more complicated numerical techniques. This procedure has great potential to be implemented in more complex systems where there are no exact solutions available except approximations.

MATHEMATICS SUBJECT CLASSIFICATION: 42C10; 35F30; 65L10; 65L60; 35K20

1. Introduction

B-polynomials (Bernstein polynomials basis) were originally introduced in the approximation of continuous functions $f(x)$ on an interval $[a, b]$ (see [1]),

$$B_{n,i}(x) = \binom{n}{i} \frac{(b-x)^{n-i}(x-a)^i}{(b-a)^n}, \quad i = 0, 1, \ldots, n,$$

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which form a basis for the space of algebraic polynomials of degree less than or equal to n, but they are not orthogonal. These basis provides a demonstration of the theorem of Weierstrass that a continuous function on an interval may be approximated to any specified tolerance by polynomials of sufficiently high degree (see [1,2]). It is a widely used basis due to the good properties like the recursive relation, the symmetric properties, and making partition of unity (see, for instance [3–7]). These properties make them the most commonly used basis in approximation theory and computer aided geometric design (CAGD) [1,8,9].

The properties of **B**-polynomials offer valuable insight into its geometrical behavior and has won widespread acceptance as the basis for Bézier curves and surfaces in CAGD [8]. In CAGD, it is often necessary to obtain polynomial approximations to more complicated functions, defined over finite domains. The approximation scheme must often incorporate certain essential features-such as interpolation of boundary values and/or derivatives to a specified order; guaranteed convergence as the degree of the approximant is increased; satisfaction of prescribed bounds on the approximation error; computational efficiency and numerical stability; and output of results in a form compatible with CAGD conventions (e.g., the Bernstein-Bézier representation). **B**-polynomials are incredibly useful mathematical tools as they are precisely defined, calculated rapidly on a modern computer system, and can be differentiated and integrated without difficulty.

There exists in the literature a number of approximate methods for solving numerically various classes of differential equations by using orthogonal polynomials (see, for instance, [10–16]). A great interest to use **B**-polynomials for solving different classes of differential equations (see, for instance, [17,18]). Farouki and GoodMan [19] demonstrate that **B** -polynomials have optimal stability in Γ_n (the set of nonnegative bases for the space of all polynomials of degree n on the interval $[a, b]$), in the sense that there exists no other nonnegative basis that gives systematically smaller condition numbers than it. Also, they stated that although it is not uniquely optimal, no other basis in common use enjoys this distinction, and it is uncertain whether other conceivable optimally-stable bases would share the useful properties and algorithms that we associate with the Bernstein form. In addition, there are problems for which nonpositive (e.g. Chebyshev) methods fail, so as to illustrate the importance of positivity, such problems often occur in computational Chemistry, for example.

In this paper, we introduce and implement a new numerical algorithm based on Bernstein polynomials basis and its dual for solving 2nd-order differential equations by approximating the solution in the **B**-polynomial. Motivations to interest in such polynomials are that the theoretical and numerical analysis of numerous physical and mathematical problems very often requires the expansion of an arbitrary polynomial or its derivatives and moments into a set of **B**-polynomials (see [6,20–22]). They are important in certain problems of mathematical physics; for example, in CAGD, the design of (vector-valued) functions of one or two variables, i.e., curves and surfaces, rather than the approximation of functions specified a priori. In this context, the Bézier curve and surface formulations, based on parametric polynomial representations in the Bernstein basis, provide a powerful natural language for analyzing elementary geometric characteristics such as smoothness and bounds, and performing a variety of basic geometric functions (parametric evaluation, subdivision, differentiation, intersections, etc.) through iterated sequences of linear interpolations (see [23–26]). Farouki and Rajan [6] succeeded in developing Bernstein formulations for all the basic polynomial procedures required in geometric modeling algorithms: the polynomial arithmetic operations, the substitution of polynomials, the determination of greatest common divisors, and the elimination of variables, and they remarked that these Bernstein formulations were essentially as simple as their familiar power formulations. Boyd [27] show that the Bernstein basis had an important role for testing if a polynomial of degree N in the form of a truncated Chebyshev series is free of zeros on its canonical interval.

We discuss new formulae related to **B**-polynomials which have many useful uses. New procedure to obtain numerical solutions to 2nd–order linear differential equations subject to Dirichlet conditions as linear combinations of the **B**-polynomials is presented. The procedure takes advantage of the continuity and unity partition properties of the basis set of **B**-polynomials over an interval [0, 1]. The **B**-polynomial bases vanish except the first polynomial at $x = 0$, which is equal to 1 and the last polynomial at $x = 1$, which is also equal to 1 over the interval [0, 1]. This provides greater flexibility in which to impose boundary conditions at the end points of the interval. In many applications, a matrix formulation for the **B**-polynomials of degree less than or equal to n is considered. These are straightforward to develop and are applied to solve the differential equations in this paper. The set of **B**-polynomials of degree n on an interval forms a complete basis for continuous $(n + 1)$ polynomials. In the following sections, we explain the procedure and define the polynomial basis.

2. Polynomial basis

The **B**-polynomials of nth-degree form a complete basis over [0, 1], and they are defined by

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad 0 \leqslant i \leqslant n. \tag{2.1}$$

There are $(n + 1)$ nth-degree polynomials and for convenience, we set $B_{i,\ n}(x) = 0$, if $i < 0$ or $i > n$.

A recursive definition also can be used to generate the **B**-polynomials over this interval, so that the ith nth degree **B**-polynomial can be written

$$B_{i,n}(x) = (1-x)B_{i,n-1}(x) + x\ B_{i-1,n-1}(x). \tag{2.2}$$

The derivatives of the nth degree **B**-polynomials are polynomials of degree $n - 1$ and are given by

$$DB_{i,n}(x) = n(B_{i-1,n-1}(x) - B_{i,n-1}(x)), \quad D \equiv \frac{d}{dx}. \tag{2.3}$$

Also we have

$$B_{i,n}(0) = \delta_{i,0}, \ \ B_{i,n}(1) = \delta_{i,n}. \tag{2.4}$$

Each of the **B**-polynomials is positive and also the sum of all the **B**-polynomials is unity for all real x belonging to the interval [0, 1], that is, $\sum_{i=0}^{n} B_{i,n}(x) = 1$. It can be easily shown that any given polynomial of degree n can be expanded in terms of a linear combination of the basis functions [[26], formula (2), p.3]