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ORIGINAL ARTICLE

A new Euler matrix method for solving systems of linear Volterra integral equations with variable coefficients

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Abstract This article develops an efficient solver based on collocation points for solving numerically a system of linear Volterra integral equations (VIEs) with variable coefficients. By using the Euler polynomials and the collocation points, this method transforms the system of linear VIEs into the matrix equation. The matrix equation corresponds to a system of linear equations with the unknown Euler coefficients. A small number of Euler polynomials is needed to obtain a satisfactory result. Numerical results with comparisons are given to confirm the reliability of the proposed method for solving VIEs with variable coefficients.

MSC: 45D05; 45F05; 65D30

1. Introduction

Systems of linear integral equations and their solutions are of great importance in science and engineering. Most physical problems, such as biological applications in population dynamics and genetics where impulses arise naturally or are caused by control, can be modeled by the differential equation, an integral equation or an integro-differential equation or a system of these equations. The systems of integral and integro-differential equations are usually difficult to solve analytically; so a numerical method is needed. The solution of systems of integral equations occurring in physics, biology and engineering are based on numerical methods such as the Euler–Chebyshev and Runge–Kutta methods. Recently, systems of the integral and integro-differential equations have been solved using the homotopy perturbation method [1], an efficient algorithm [2], the Lagrange method [3], the modified homotopy perturbation method [4], the rationalized Haar functions method [5,6], the differential transformation method [7], the Tau method [8], the variational iteration method [9], the Legendre matrix method [10], the Adomian method [11], the Galerkin method [12] and the Bessel matrix method [13]. In addition, solutions of systems of linear differential and integro-differential equations have been tried using the Chebyshev polynomial method and the Taylor collocation method by Sezer et al. [14,15].

In this study, in the light of the above-mentioned methods and by means of the matrix relations between the Euler poly-

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nomials, we develop a new method called the Euler matrix method for solving a system of linear VIEs with variable coefficients in the form

$$\sum_{j=1}^{k}\beta_{i,j}(x)y_j(x)=f_i(x)+\int_a^x\sum_{j=1}^{k}K_{i,j}(x,t)y_j(t)dt,$$
$$i=1,2,\dots,k,\quad 0\leqslant a\leqslant x\leqslant b, \tag{1}$$

where $y_j(x)$ is an unknown function. Also, $\beta_{i,j}(x)$, $f_i(x)$ and $K_{i,j}(x,t)$ are continuous functions defined on the interval $a\leqslant x,\quad t\leqslant b$. Moreover, the functions $K_{i,j}(x,t)$ for $i, j = 1,2,\dots,k$ can be expanded Maclaurin series.

Our aim is to find an approximate solution of (1) expressed in the truncated Euler series form

$$y_i(x)=\sum_{n=0}^{N}a_{i,n}E_n(x),\quad i=1,2,\dots,k,\quad 0\leqslant a\leqslant x\leqslant b, \tag{2}$$

so that $a_{i,n}$ for $i = 1,2,\dots,k$ and $n = 0,1,\dots,N$ are the unknown Euler coefficients, and $E_n(x)$ for $n = 0,1,\dots,N$ are the Euler polynomials of the first kind which are constructed from the following relation [16]:

$$\sum_{k=0}^{n}\binom{n}{k}E_k(x)+E_n(x)=2x^n. \tag{3}$$

For example if we take $n = 0$; $E_0(x) = 1$ and if $n = 1$; $E_1(x)=x-\frac{1}{2}$. For more details, if in Eq. (3) n varies from 0 to N we have the following linear matrix equation:

$$\underbrace{\frac{1}{2}\begin{pmatrix}2&0&0&\cdots&0\\ \binom{1}{0}&2&0&\cdots&0\\ \binom{2}{0}&\binom{2}{1}&2&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ \binom{N}{0}&\binom{N}{1}&\binom{N}{2}&\cdots&2\end{pmatrix}}_{\widehat{D}}\underbrace{\begin{pmatrix}E_0(x)\\E_1(x)\\E_2(x)\\ \vdots\\E_N(x)\end{pmatrix}}_{E^T(x)}=\underbrace{\begin{pmatrix}1\\x\\x^2\\ \vdots\\x^N\end{pmatrix}}_{X^T(x)}. \tag{4}$$

Since $\widehat{D}$ is a lower triangular matrix with nonzero diagonal elements, it is nonsingular and hence D^{-1} exists. Thus, the Euler vector can be given directly from

$$E^T(x)=\widehat{D}^{-1}X^T(x). \tag{5}$$

This paper is organized as follows. In Section 2, we recall the basic concepts of Euler polynomials and their relevant properties needed hereafter. In Section 3, the way of constructing the proposed matrix method for solving VIEs is described using the above-mentioned polynomials. In Section 4, accuracy of the solution and error analysis are briefly discussed. In Section 5, the proposed method is applied to several types of linear VIEs with variable coefficients, and comparisons are made with the existing analytic or numerical solutions that were reported in other published works in the literature. The paper is closed in Section 6, by concluding and giving ideas for future work.

2. Euler polynomials

Euler polynomials that considered in this paper have special properties and applications in different fields of mathematics. Euler polynomials and numbers (introduced by Euler in 1740) also possess an extensive literature and several interesting applications in Number Theory (see, for instance, [17,18]). In many respects, they are closely related to the theory of Bernoulli polynomials and numbers. Numerous interesting (and useful) properties and relationships involving these polynomials and numbers can be found in many books and tables on this subject (see, for example, [19–22]).

Euler polynomials satisfies the following interesting property

Property 1 [16]. *$E_n(x+1)+E_n(x)=2x^n$.*

Property 2 [16]. *$E'_n(x)=nE_{n-1}(x),\quad n=1,2,\dots$*

Property 3 ([23–26]). *$E_n(x)=\frac{1}{n+1}\sum_{k=1}^{n+1}(2-2^{k+1})\binom{n+1}{k}B_k(0)x^{n+1-k}$, where $B_k(x)$, $k = 0,1,\dots$, are the Bernoulli polynomials of order k which can be defined by*

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k(x)=(n+1)x^n.$$

From the third property, the Euler polynomials can be represented in the following form

$$\underbrace{\begin{pmatrix}E_0(x)\\E_1(x)\\ \vdots\\E_N(x)\end{pmatrix}}_{E^T(x)}=\underbrace{\begin{pmatrix}\frac{(2-2^2)}{1}\binom{1}{1}B_1(0)&0&\cdots&0\\ \frac{(2-2^3)}{2}\binom{2}{2}B_2(0)&\frac{(2-2^2)}{2}\binom{2}{1}B_1(0)&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ \frac{(2-2^{N+2})}{N}\binom{N}{N}B_N(0)&\frac{(2-2^{N+1})}{N}\binom{N}{N-1}B_{N-1}(0)&\cdots&\frac{(2-2^2)}{N}\binom{N}{1}B_1(0)\end{pmatrix}}_{D}\underbrace{\begin{pmatrix}1\\x\\ \vdots\\x^N\end{pmatrix}}_{X^T(x)}. \tag{6}$$

In comparison to (7), we must have $D=\widehat{D}^{-1}$. Thus, the Euler vector can be given directly from

$$E^T(x)=DX^T(x)\Longleftrightarrow E(x)=X(x)D^T. \tag{7}$$

3. Euler matrix method for solving VIEs

In this section, we convert Eq. (1) to linear systems of matrix equations which can be easily solved. Let us show Eq. (1) in the form

$$\beta(x)\mathbf{y}(x)=\mathbf{f}(x)+\mathbf{V}(x), \tag{8}$$

where

$$\beta(x)=\begin{bmatrix}\beta_{1,1}(x)&\beta_{1,2}(x)&\cdots&\beta_{1,k}(x)\\ \beta_{2,1}(x)&\beta_{2,2}(x)&\cdots&\beta_{2,k}(x)\\ \vdots&\vdots&\ddots&\vdots\\ \beta_{k,1}(x)&\beta_{k,2}(x)&\cdots&\beta_{k,k}(x)\end{bmatrix},\quad \mathbf{y}(x)=\begin{bmatrix}y_1(x)\\y_2(x)\\ \vdots\\y_k(x)\end{bmatrix},\quad \mathbf{f}(x)=\begin{bmatrix}f_1(x)\\f_2(x)\\ \vdots\\f_k(x)\end{bmatrix},$$

$$\mathbf{V}(x)=\int_a^x\mathbf{K}(x,t)\mathbf{y}(t)dt,\quad \mathbf{K}(x,t)=\begin{bmatrix}K_{1,1}(x,t)&K_{1,2}(x,t)&\cdots&K_{1,k}(x,t)\\ K_{2,1}(x,t)&K_{2,2}(x,t)&\cdots&K_{2,k}(x,t)\\ \vdots&\vdots&\ddots&\vdots\\ K_{k,1}(x,t)&K_{k,2}(x,t)&\cdots&K_{k,k}(x,t)\end{bmatrix},$$

$$\mathbf{V}(x)=\begin{bmatrix}V_1(x)\\V_2(x)\\ \vdots\\V_k(x)\end{bmatrix}, \tag{9}$$