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ORIGINAL ARTICLE

A note on paranormed difference sequence spaces of fractional order and their matrix transformations

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Abstract In the present paper, new classes of difference sequence spaces $X(\Gamma, \Delta^\alpha, u, p)$ for $X \in \{\ell_\infty, c, c_0\}$ are introduced by using the fractional difference operator Δ^α, defined by $\Delta^\alpha(x_k) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}$, where (u_n) is a sequence satisfying certain conditions and α is a proper fraction. In fact, the operator Δ^α generalizes the difference operators used in several difference sequence spaces such as $X(\Delta)$, $X(\Delta^2)$, $\Delta^m(X)$, $\Delta X(p)$, $\Delta^m X(p)$, $X(u; \Delta^2)$, $X(u, \Delta, p)$, $X(u, \Delta^2, p)$ (see [3–12,20–22]). Also, we investigate the topological structures and establish $\alpha-$, $\beta-$ and $\gamma-$ duals of the spaces $X(\Gamma, \Delta^\alpha, u, p)$. Furthermore, the matrix transformations between these spaces and the basic sequence spaces $\ell_\infty(q)$, $c_0(q)$ and $c(q)$ are characterized.

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1. Introduction, preliminaries, and definitions

Let $\Gamma(m)$ be the Gamma function of a real number m and $m \notin \{0, -1, -2, -3, \ldots\}$. By the definition, it can be expressed as an improper integral, i.e.

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} dt. \tag{1.1}$$

From the Eq. (1.1), it is clear that if $m \in \mathbb{N}_0$, the set of nonnegative integers, then $\Gamma(m+1) = m!$ For this reason, Gamma function is regarded as the generalization of elementary factorial function. Now, we state some properties of Gamma function which are essential throughout the text.

(i) For any natural number m, $\Gamma(m+1) = m!$.
(ii) For any real number m and $m \notin \{0, -1, -2, -3, \ldots\}$, $\Gamma(m+1) = m\Gamma(m)$.
(iii) For particular cases, we have $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = 2!$, $\Gamma(4) = 3!$, ...

Let w be the space all real valued sequences. Any subspace of w is called a *sequence space*. By ℓ_∞, c and c_0, we denote the spaces of all bounded, convergent and null sequences, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. Also by ℓ_1 and ℓ_p,

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we write the spaces of all absolutely and p-absolutely summable series, normed by $\sum_k|x_k|$ and $(\sum_k|x_k|^p)^{1/p}$ for $1 < p < \infty$, respectively.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $M = \max(1, \sup_k p_k)$. The linear spaces $\ell_\infty(p)$, $c_0(p)$, $c(p)$ and $\ell(p)$ defined by Maddox [1,2] are as follows:

$$\ell_\infty(p) = \{x = (x_k) \in w : \sup_k |x_k|^{p_k} < \infty\},$$

$$c_0(p) = \{x = (x_k) \in w : \lim_{k\to\infty} |x_k|^{p_k} = 0\},$$

$$c(p) = \{x = (x_k) \in w : \lim_{k\to\infty} |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C}\}$$

and

$$\ell(p) = \{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\}.$$

These are complete spaces paranormed by

$$h_1(x) = \sup_k |x_k|^{p_k/M} \text{ if and only if } \inf_k p_k > 0 \text{ and } h_2(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/M} \text{ respectively.}$$

Let $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk} where $n, k \in \mathbb{N}_0$. For the sequence spaces X and Y we write a matrix mapping A: $X \to Y$ defined by

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}_0). \tag{1.2}$$

For every $x = (x_k) \in X$, we call Ax as the A-transform of x if the series $\sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}_0$. By (X,Y), we denote the class of all infinite matrices A such that A: $X \to Y$. Thus, $A \in (X,Y)$ if and only if the series in (1.2) converges for each $n \in \mathbb{N}_0$.

By bs and cs, we write the spaces of all bounded and convergent series, respectively. Now with the help of matrix transformations, define the set $S(X, Y)$ by

$$S(X, Y) = \{z = (z_k) : xz = (x_k z_k) \in Y \text{ for all } x \in X\}. \tag{1.3}$$

With the notation of (1.3), we redefine the $\alpha-$, $\beta-$ and $\gamma-$ duals of a sequence space X, respectively, as follows:

$$X^\alpha = S(X, \ell_1), \quad X^\beta = S(X, cs) \text{ and } X^\gamma = S(X, bs).$$

2. The sequence spaces $X(\Gamma, \Delta^\alpha, u, p)$ for $X \in \{\ell_\infty, c, c_0\}$

In this section, we define the classes of new sequence spaces $\ell_\infty(\Gamma, \Delta^\alpha, u, p)$, $c(\Gamma, \Delta^\alpha, u, p)$ and $c_0(\Gamma, \Delta^\alpha, u, p)$ by using the difference operator Δ^α. We also obtain certain results concerning their generalizations of other difference operators as well as their linear and topological properties.

Let $\mathcal{U}$ be the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for all $n \in \mathbb{N}_0$. For $u \in \mathcal{U}$, let $1/u = (1/u_n)$. For a proper fraction α and $X \in \{\ell_\infty, c, c_0\}$, we define the sequence spaces $X(\Gamma, \Delta^\alpha, u, p)$ by

$$X(\Gamma, \Delta^\alpha, u, p) = \left\{x = (x_k) \in w : y_k = \left(\sum_{j=0}^{k} u_j \Delta^\alpha x_j\right) \in X(p)\right\}, \tag{2.1}$$

where $\Delta^\alpha x_j$ is defined as follows:

$$\Delta^\alpha(x_j) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)} x_{j+i}. \tag{2.2}$$

In particular, one can observe that

- $\Delta^{\frac{1}{2}}x_k = x_k - \frac{1}{2}x_{k+1} - \frac{1}{8}x_{k+2} - \frac{1}{16}x_{k+3} - \frac{5}{128}x_{k+4} - \frac{7}{256}x_{k+5} - \frac{21}{1024}x_{k+6} + \cdots$
- $\Delta^{-\frac{1}{2}}x_k = x_k + \frac{1}{2}x_{k+1} + \frac{3}{8}x_{k+2} + \frac{5}{16}x_{k+3} + \frac{35}{128}x_{k+4} + \frac{63}{256}x_{k+5} + \frac{231}{1024}x_{k+6} + \cdots$
- $\Delta^{\frac{1}{3}}x_k = x_k - \frac{1}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{5}{81}x_{k+3} - \frac{10}{243}x_{k+4} - \frac{22}{729}x_{k+5} - \frac{154}{6561}x_{k+6} + \cdots$
- $\Delta^{\frac{2}{3}}x_k = x_k - \frac{2}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{4}{81}x_{k+3} - \frac{7}{243}x_{k+4} - \frac{14}{729}x_{k+5} - \frac{91}{6561}x_{k+6} + \cdots$
- $\Delta^{\frac{1}{6}}x_k = x_k - \frac{1}{6}x_{k+1} - \frac{5}{72}x_{k+2} - \frac{55}{1296}x_{k+3} - \frac{935}{31104}x_{k+4} - \frac{21505}{933120}x_{k+5} + \cdots$

Throughout the text, we assume that the series defined in (2.2) is convergent for all $x \in X$. Moreover, if $\alpha = m$, a positive integer, then the infinite sum defined in (2.2) reduces to a finite sum, i.e., $\sum_{i=0}^{m}(-1)^i \frac{\Gamma(m+1)}{i!\Gamma(m-i+1)} x_{j+i}$. Subsequently, for $\alpha > 0$, $\Delta^{-\alpha}(x_j) = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(1-\alpha)}{i!\Gamma(1-\alpha-i)} x_{j+i}$.

In fact, this operator generalizes several difference operators introduced by Kızmaz [3], Et [4], Et and Çolak [5], Ahmad and Mursaleen [6], Et and Basarir [7] and many others (see [8–19]). Again under the suitable conditions, new classes defined in (2.1) generalize many known spaces defined by Mursaleen [20], Asma and Çolak [21], Bektas [22], Baliarsingh [23] and others (see [24–28]).

Now using notation (1.2), define the sequence $y = (y_k)$, which can be frequently used as the $\Gamma(\Delta, \alpha, u)$-transform of a sequence $x = (x_k)$, where $\Gamma(\Delta, \alpha, u) = (\gamma^\alpha_{nk})$ is an infinite matrix and

$$\gamma^\alpha_{nk} = \begin{cases} \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha+1-i)} u_{k-i}, & (k \geqslant n), \\ 0, & (k < n). \end{cases} \tag{2.3}$$

Throughout the text, we use the convention that any term with negative subscript is equal to zero.

Now, we give some interesting results of these spaces concerning their topological structures. In order to avoid the repetition of the similar statements, we give the proof of only one from these three spaces. The proofs of other spaces may be obtained by using similar arguments.

Theorem 1. *The sequence spaces $X(\Gamma, \Delta^\alpha, u, p)$ for $X \in \{\ell_\infty, c, c_0\}$ are linear metric spaces paranormed by g, defined by*

$$g(x) = \sup_k \left|\sum_{j=0}^{k} u_j \Delta^\alpha x_j\right|^{p_k/M}. \tag{2.4}$$

Proof. We prove the theorem for the space $\ell_\infty(\Gamma, \Delta^\alpha, u, p)$. It is clear that $g(\theta) = 0$ and $g(-x) = g(x)$ for all $x \in \ell_\infty(\Gamma, \Delta^\alpha, u, p)$. For linearity of $\ell_\infty(\Gamma, \Delta^\alpha, u, p)$ with respect to coordinate wise addition and scalar multiplication, we take any two sequences s, $t \in \ell_\infty(\Gamma, \Delta^\alpha, u, p)$ and scalars $\alpha', \beta' \in \mathbb{R}$. Since the operator Δ^α is linear and by Maddox [29], we obtain that

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