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ORIGINAL ARTICLE

Two new forms of half-discrete Hilbert inequality

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Abstract In this paper, we introduce two new forms of the half-discrete Hilbert inequality. The first form is a sharper form of the half-discrete Hilbert inequality and is related to Hardy inequality. In the second one, we give a differential form of this inequality.

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1. Introduction

If $f(x), g(y) > 0,\ 0 < \int_0^\infty f^p(x)dx < \infty$, and $0 < \int_0^\infty g^q(y)dy < \infty$, then the Hardy-Hilbert's inequality may be written as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}, \tag{1.1}$$

where the constant $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible [1].

Recently, many generalizations of (1.1) were given. Yang et al. [2] obtained the following extension of (1.1) as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left\{\int_0^\infty x^{p-1-\lambda}f^p(x)dt\right\}^{\frac{1}{p}} \times \left\{\int_0^\infty y^{q-1-\lambda}g^q(y)dt\right\}^{\frac{1}{q}}, \tag{1.2}$$

where $\lambda > 0$ and the constant $B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$ (the Beta function) is the best possible. The following general inequality was given in [3]

$$\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy < k(pA_2)\left\{\int_0^\infty x^{pqA_1-1}f^p(x)dx\right\}^{\frac{1}{p}} \times \left\{\int_0^\infty y^{pqA_2-1}g^q(x)dx\right\}^{\frac{1}{q}},$$

where $k(pA_2) = \int_0^\infty K(1,t)t^{-pA_2}$ is the best possible constant, $K(x, y) \geqslant 0$ is a homogeneous function of degree $-\lambda(\lambda > 0)$, $A_1 \in \left(\frac{1-\lambda}{q},\frac{1}{q}\right)$, $A_2 \in \left(\frac{1-\lambda}{p},\frac{1}{p}\right)$ and $pA_2 + qA_1 = 2 - \lambda$. In [4] the following two new forms of (1.1) were proved:

For $f, g > 0$, $f, g \in L(0, \infty)$, define $F(x) = \int_0^x f(u)du$ and $G(x) = \int_0^x g(u)du$, then for $\lambda > 0$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \leqslant \frac{\lambda^2}{pq}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)\left(\int_0^\infty x^{-\lambda-1}F^p(x)dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty y^{-\lambda-1}G^q(y)dy\right)^{\frac{1}{q}}. \tag{1.3}$$

For $\lambda > n\max(p, q)$, $n = 0, 1, \ldots$, and assuming that f, g satisfy the conditions of Lemma 2.1 (see Section 2.2), then:

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$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \leqslant \frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)} \times \left(\int_0^\infty x^{p(n+1)-\lambda-1}(f^{(n)}(x))^p dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty y^{q(n+1)-\lambda-1}(g^{(n)}(y))^q dy\right)^{\frac{1}{q}}, \quad (1.4)$$

where the constant factors in both (1.3) and (1.4) are the best possible.

Refinements of some Hilbert-type inequalities by virtue of various methods are obtained in [5–7]. A survey of some recent results concerning Hilbert and Hilbert-type inequalities can be found in [8].

In [9] Yang introduced the following half-discrete Hilbert's inequality

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} dx < B(\lambda_1,\lambda_2)\left(\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)dx\right)^{\frac{1}{p}} \times \left(\sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q\right)^{\frac{1}{q}}, \quad (1.5)$$

here, $\lambda_1,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = \lambda,\ 0 < \lambda_1 < 1$, and the constant $B(\lambda_1,\ \lambda_2)$ is the best possible. In particular if we set $\lambda_1 = \frac{\lambda}{p}, \lambda_2 = \frac{\lambda}{q}$, we get from (1.5)

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} dx < B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left(\int_0^\infty x^{p-\lambda-1} f^p(x)dx\right)^{\frac{1}{p}} \times \left(\sum_{n=1}^\infty n^{q-\lambda-1} a_n^q\right)^{\frac{1}{q}}. \quad (1.6)$$

For extensions and other half-discrete Hilbert's inequalities see for example [10,11].

If $p > 1$, $f(x) > 0$, and $F(x) = \int_0^x f(t)dt$, then the famous Hardy inequality [1] is given as

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad (1.7)$$

the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. A weighted form of (1.3) is *given* also by Hardy [1] as

$$\int_0^\infty x^a\left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1-a}\right)^p \int_0^\infty x^a f^p(x)dx, \quad (1.8)$$

where $a < p-1$ and the constant $\left(\frac{p}{p-1-a}\right)^p$ is the best possible. Inequality (1.7) was discovered by Hardy while he was trying to introduce a simple proof of Hilbert inequality. For more information about inequalities (1.7), (1.8) and their history and development, we refer the reader to the papers [12,13].

In this paper by estimating $\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx$, we introduce two new inequalities with a best constant factor, similar to (1.3) and (1.4), the first one contained in Theorem 3.1 gives a relation between Hardy inequality and half-discrete Hilbert inequality, the second inequality contained in Theorem 3.2 gives a differential form of half-discrete Hilbert inequality.

2. Preliminaries and Lemmas

Recall that the Gamma function $\Gamma(\theta)$ and the Beta function $B(\mu,\ \nu)$ are defined, respectively, by

$$\Gamma(\theta) = \int_0^\infty t^{\theta-1}e^{-t}dt, \quad \theta > 0,$$

$$B(\mu,\nu) = \int_0^\infty \frac{t^{\mu-1}}{(t+1)^{\mu+\nu}}dt, \quad \mu,\nu > 0.$$

By the definition of the Gamma function, the following equality holds

$$\frac{1}{(x+y)^\lambda} = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-(x+y)t}dt. \quad (2.1)$$

We will need the following three Lemmas (Lemmas 2.1 and 2.2 are given in [4]):

Lemma 2.1. *Let* $r > 1,\ \frac{1}{r}+\frac{1}{s} = 1,\ \varphi > 0,\ \varphi \in L(0,\infty),\ \Phi(x) = \int_0^x \varphi(u)du$, *then for* $t,\ \alpha > 0$ *we have*

$$\int_0^\infty e^{-tx}\varphi(x)dx \leqslant t^{\frac{1}{s}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left\{\int_0^\infty x^{-\alpha r}e^{-tx}\Phi^r(x)dx\right\}^{\frac{1}{r}}.$$

Lemma 2.2. *Let* $r > 1,\ \frac{1}{r}+\frac{1}{s} = 1,\ \varphi > 0$, *the derivatives* $\varphi',\ \varphi''$, *…,* $\varphi^{(k)}$ *exists and positive and* $\varphi^{(k)} \in L(0,\ \infty)(k = 0, 1, \ldots)$ $(\varphi^{(0)}{:=}\varphi)$, *moreover, suppose that* $\varphi(0) = \varphi'(0) = \cdots = \varphi^{(k-1)}(0) = 0$, *then for* $t,\ \alpha > 0$ *we have*

$$\int_0^\infty e^{-tx}\varphi(x)dx \leqslant t^{-k-\frac{1}{s}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left\{\int_0^\infty x^{-\alpha r}e^{-tx}\left(\varphi^{(k)}(x)\right)^r dx\right\}^{\frac{1}{r}}.$$

Lemma 2.3. *Let* $r > 1,\ \frac{1}{r}+\frac{1}{s} = 1,\ a_n > 0$, *then for* $t > 0$ *and* $0 \leqslant \beta < \frac{1}{r}$ *we have*

$$\sum_{n=1}^\infty e^{-nt}a_n < t^{\beta-\frac{1}{r}}\Gamma(1-\beta r)^{\frac{1}{r}}\left\{\sum_{n=1}^\infty n^{\beta s}e^{-nt}a_n^s\right\}^{\frac{1}{s}}.$$

Proof. Using Hölder's inequality, we get

$$\sum_{n=1}^\infty e^{-nt}a_n = \sum_{n=1}^\infty \left\{n^{-\beta}e^{-\frac{nt}{r}}\right\}\left\{n^\beta e^{-\frac{nt}{s}}a_n\right\}$$
$$< \left(\sum_{n=1}^\infty n^{-\beta r}e^{-nt}\right)^{\frac{1}{r}}\left(\sum_{n=1}^\infty n^{\beta s}e^{-nt}a_n^s\right)^{\frac{1}{s}}$$
$$< \left(\int_0^\infty x^{-\beta r}e^{-tx}dx\right)^{\frac{1}{r}}\left(\sum_{n=1}^\infty n^{\beta s}e^{-nt}a_n^s\right)^{\frac{1}{s}}$$
$$= t^{\beta-\frac{1}{r}}\Gamma(1-\beta r)^{\frac{1}{r}}\left\{\sum_{n=1}^\infty n^{\beta s}e^{-nt}a_n^s\right\}^{\frac{1}{s}}. \qquad \square$$

3. Main results

In this section, we introduce the main two results in this paper. Theorem 3.1 gives a new form of the half-discrete Hilbert inequality (1.6) which is related to the famous Hardy inequality. In Theorem 3.2, we introduce another new form of the half-discrete Hilbert inequality, namely a differential form which is an extension of (1.6). Both of the obtained inequalities are with a best constant factor.