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A note on the sensitivity of the strategic asset allocation problem

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ABSTRACT

The Markowitz mean–variance portfolio optimization problem is a quadratic programming problem whose first-order conditions require the solution of a linear system. It is well known that the optimal portfolio weights are sensitive to parameter estimates, particularly the mean return vector. This has generally been attributed to the interaction of estimation error and optimization. In this paper we present some examples that suggest the linear system produced by the first-order conditions is ill-conditioned and it is this property that gives rise to the sensitivity of the optimal weights.

1. Introduction

The mean–variance portfolio optimization problem dates to the pioneering work of Markowitz [1]. It is well known that the optimal portfolio weights are sensitive to the parameter input, particularly the mean return vector. See, for the example, the work of Best and Grauer [2], Broadie [3], Chopra [4], Chopra and Ziemba [5], Frankfurter et al. [6], and Michaud [7]. This sensitivity has generally been attributed to the tendency for optimization to magnify the effects of estimation error. For this reason, Michaud [7] has referred to "portfolio optimization" as "error maximization".

There is now a vast literature on how to deal with the problem. As might be expected, the literature focuses on improved estimation procedures and model variations. Efforts to improve parameter estimation procedures include the work of Jobson and Korkie [8] and Jorion [9,10] on shrinkage estimators and Ledoit and Wolf [11] on reducing the error in the estimation of the covariance matrix. A host of researchers look at robust portfolio optimization (see, for example, Goldfarb and Iyengar [12], Garlappi et al. [13], and Lu [14]). Different formulations of the problem also include the work of Black and Litterman [15], Konno and Yamazaki [16], Simaan [17], and more recently, Jangannathan and Ma [18] and DeMiguel et al. [19]. It is important to clarify what financial theorists mean when they refer to robust portfolio optimization. In the general sense, robust optimization implies finding solutions that can be modified later in an effective manner once actual conditions are known. However, with portfolio management, the input parameters are consistently changing, and robustness in this setting refers to finding solutions that are insensitive to these changes. That is, portfolio formation strategies are sought that are relatively immune to variations in input values. Here, we offer no solution to the problem. Rather, we make a simple but important point. It is not always true that optimization magnifies estimation error and we show this using the basic Economic Order Quantity Model. The implication is that there has to be a deeper explanation of why portfolio weights are so sensitive to estimation error. In our view, this explanation has to do with *the underlying structure of the model*. We argue that the first-order conditions of the Markowitz portfolio optimization model result in a linear system that is ill-conditioned and *it is this poor conditioning that leads to the extreme sensitivity of the portfolio weights*.

This observation has a number of important implications. First, it is very unlikely that improved estimation techniques will solve the problem. DeMiguel et al. [20] took an exhaustive look at how existing improved estimation procedures and different models stacked up against a naive $1/n$ portfolio (a portfolio with funds divided equally among n assets). They found that none of these offered any significant performance improvement based on standard measures (including the Sharpe ratio and certainty-equivalent return). This is consistent with our observation on conditioning that we are not likely to find a magic bullet to solve the problem.

The work of Ledoit and Wolf [11] is particularly interesting. They consider only the covariance matrix. They offer the equivalent of a Stein estimator (i.e. a shrinkage estimator) for the covariance matrix and show that it has nice consistency properties as the dimension of the problem (both in the dimension of the covariance matrix and the dimension of the data set used to estimate it) gets large. Their estimator performs reasonably well and the condition of the covariance matrix falls dramatically for problems with a large number of assets and a large dataset. However, there are problems of interest where there is no guarantee that the dimension of the covariance matrix will be high. We have in mind the

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Strategic Asset Allocation Problem (SAAP) where only a handful of global asset classes is considered. An example is the problem considered by Black and Litterman [15].

2. It is not just optimization magnifying estimation error

Among others, Michaud [7] has argued that the sensitivity of the portfolio optimization problem is due to optimization magnifying estimation error. But this cannot be the complete explanation. After all there are many examples of optimization problems where slight errors in the parameter input are not magnified by optimization. For instance consider the Economic Order Quantity model used for inventory decisions. This model argues that the order size that minimizes transaction costs is

$$x^* = \sqrt{kD} \tag{1}$$

where x^* is the order size, k is a parameter that depends on the costs of holding and processing inventory, and D is the demand rate for the inventory over the period under consideration. Now suppose a small error ε is made when D is estimated. Then we have that

$$x^*(\varepsilon) = \sqrt{kD(1+\varepsilon)} \tag{2}$$

and the relative error is

$$\frac{x^*(\varepsilon) - x^*}{x^*} = \frac{\sqrt{kD(1+\varepsilon)} - \sqrt{kD}}{\sqrt{kD}} \simeq \varepsilon/2. \tag{3}$$

Thus, a 10% error in the estimation of demand, leads to approximately a 5% change in the recommended inventory level. The implication is that an explanation of solution sensitivity *must appeal to the underlying structure of the problem.*

3. The sensitivity of the SAA portfolio

Here is an example which demonstrates the sensitivity of the SAAP. Suppose an investor is considering a portfolio of three funds: LargeCap Equity; Foreign; and Bond. The investor estimates that expected returns for these are:

Asset	**Return**
LargeCap	0.1213
Foreign	0.1548
Bond	0.0923

(4)

and the covariance matrix is

	LargeCap	*Foreign*	*Bond*
LargeCap	0.02528	0.02098	0.00411
Foreign	0.02098	0.05452	0.00085
Bond	0.00411	0.00085	0.00487.

(5)

Let the random return of asset i, r_i, be normally distributed with mean $\bar{r}_i$ and variance σ_i^2. Let the covariance of the returns on assets i and j be c_{ij}. Suppose the investor is considering a portfolio where a proportion, x_i, of his total investment will go into asset i. The expected return on the portfolio is

$$\bar{r}_p = \bar{r}_1 x_1 + \bar{r}_2 x_2 + \bar{r}_3 x_3 \tag{6}$$

and, for convenience, we define the risk measure to be 1/2 the variance of the portfolio return:

$$\Phi(x_1, x_2, x_3) = \frac{1}{2} Var(r_p) \tag{7}$$

$$= \frac{1}{2} x^T \Omega x \tag{8}$$

$$= \frac{1}{2}[\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2 + 2c_{12}x_1x_2 + 2c_{13}x_1x_3 + 2c_{23}x_2x_3] \tag{9}$$

where $r_p = r_1x_1 + r_2x_2 + r_3x_3$ is the uncertain portfolio return, Ω is the covariance matrix, and $x = (x_1, x_2, x_3)^T$ is a vector of portfolio weights. This investor wishes to minimize the variance of his portfolio return subject to it producing a mean return r_0. Hence he will solve the following SAAP:

$$\begin{aligned} \min \quad & \Phi(x_1, x_2, x_3) \\ \text{s.t.} \quad & \bar{r}_1 x_1 + \bar{r}_2 x_2 + \bar{r}_3 x_3 = r_0 \\ & x_1 + x_2 + x_3 = 1. \end{aligned} \tag{10}$$

The first-order necessary conditions require a solution of a linear system:

$$\begin{bmatrix} \sigma_1^2 & c_{12} & c_{13} & -\bar{r}_1 & -1 \\ c_{12} & \sigma_2^2 & c_{23} & -\bar{r}_2 & -1 \\ c_{13} & c_{23} & \sigma_3^2 & -\bar{r}_3 & -1 \\ \bar{r}_1 & \bar{r}_2 & \bar{r}_3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_r \\ \lambda_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ r_0 \\ 1 \end{pmatrix} \tag{11}$$

where λ_r and λ_x are Lagrange multipliers. The matrix of this system is called the *augmented covariance matrix* and we represent it with Ω_+.

The solution of the system with $r_0 = 0.135$ and the parameter input described in (4) and (5) is:

$$x_1 = 0.195, \quad x_2 = 0.593, \quad x_3 = 0.212. \tag{12}$$

Suppose now the expected return on the LargeCap asset class is changed from 12.13% to 13.34%, a change of 10%. Then the new portfolio weights are

$$x_1 = 0.503, \quad x_2 = 0.352, \quad x_3 = 0.145. \tag{13}$$

Note that the LargeCap weight, x_1, increases by 160%; the other two change by an average of 36%. So a small change in a single input parameter can give rise to substantial changes in the optimal portfolio weights.

4. The origin of the sensitivity

Consider the linear system

$$Ax = b \tag{14}$$

where A is an $n \times n$ matrix, and x and b are $n \times 1$ vectors. We assume that A is nonsingular. One definition of the *norm* of the matrix A is

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \tag{15}$$

where

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{16}$$

is the usual vector norm. It is easy to show that the definition (15) is equivalent to

$$\|A\| = \max_{\|x\|=1} \|Ax\|. \tag{17}$$

The *condition* of A, $\kappa(A)$, is defined as

$$\kappa(A) = \|A\| \times \left\|A^{-1}\right\|. \tag{18}$$

Suppose the matrix is perturbed from A to $A + \delta A$. This leads to the perturbed solution $x + \delta x$ so that

$$(A + \delta A)(x + \delta x) = b. \tag{19}$$

Hence we have two systems

$$(A + \delta A)(x + \delta x) = b \tag{20}$$

$$Ax = b. \tag{21}$$