Procedia Computer Science
Volume 80, 2016, Pages 233–244
ICCS 2016. The International Conference on Computational Science

A Cylindrical Basis Function for Solving Partial Differential Equations on Manifolds

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Abstract
Numerical solution of partial differential equations (PDEs) on manifolds continues to generate a lot of interest among scientists in the natural and applied sciences. On the other hand, recent developments of 3D scanning and computer vision technologies have produced a large number of 3D surface models represented as point clouds. Herein, we develop a simple and efficient method for solving PDEs on closed surfaces represented as point clouds. By projecting the radial vector of standard radial basis function(RBF) kernels onto the local tangent plane, we are able to produce a representation of functions that permits the replacement of surface differential operators with their Cartesian equivalent. We demonstrate, numerically, the efficiency of the method in discretizing the Laplace Beltrami operator.

Keywords: Radial basis function, closed manifolds, large point clouds, Laplace-Beltrami operator, partial differential equations

1 Introduction

Many applications in the natural and applied sciences require the solution of partial differential equations on manifolds. Such applications arise in areas such as computer graphics[18, 10, 1], image processing[20, 2, 9, 21], mathematical physics[5], biological systems[17, 8], and fluid dynamics[11, 12, 19].A lot more interest, especially in computer graphics, has been generated around solution of PDEs on closed 2D manifolds, as these arise as boundaries of 3D objects. The development of high resolution 3D scanning devices, which capture these surfaces as point clouds, have made numerical methods that can be applied directly to point clouds very attractive.

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Selection and peer-review under responsibility of the Scientific Programme Committee of ICCS 2016

doi:10.1016/j.procs.2016.05.316

A class of numerical methods that have been developed to solve PDEs on closed surfaces, involve the expression of differential operators as projections of their Cartesian equivalents onto local tangent planes via a projection operator $(I - \vec{n}\vec{n}^T)$. The resulting operators are then discretized using, for example, finite element methods[6]. Recently, the projection method has been extended to PDEs defined on manifolds represented as point clouds [4]. For such methods, functions defined on the manifold are represented using radial basis functions (RBF). The surface differential operators are obtained by applying a projection operator to the RBF discretization of the Cartesian equivalent.

While the projection methods modify Cartesian differential operators, another class of methods embed the surface PDE into $\Re^3$ so that solutions to the embedded problem when restricted to the surface provide the solution on the surface[7]. Because these methods result in embeded PDEs posed in $\Re^3$, spatial complications arise when the PDEs have to be solved on a restricted surface domain. The closest point method, developed in [16], attempts to resolve this problem by extending the functions into $\Re^3$ in a way that makes them constant in the normal direction. This allows the simple replacement of surface differential operator by their Cartesian equivalent. The method however requires a high order interpolation at each time step in order to obtain solutions on the surface.

Recently, the orthogonal gradient method presented in[13], was introduced to extend this idea to point clouds. Here, $2N$ additional nodes are introduced during the construction of a distance function, to force the function defined on the surface to be constant in the normal direction. The $2N$ nodes are chosen according to an offset parameter δ which controls their distance from the surface. The accuracy of the method and condition number of the resulting differential matrix is however sensitive to the choice of δ. The use of $2N$ additional nodes, to enforce derivative constraints, increases the computational complexity of forming the interpolation and differential matrices.

In this work we propose a modified RBF kernel, the Cylindrical Basis Function (CBF), which is intrinsically constant in the normal direction at each point of a surface. The modified kernel, when used in the representation of smooth functions defined on a closed surface, allows surface differential operators to be replaced by their Cartesian equivalent without the need to impose additional constraints on the function or have an implicit representation of the surface. We also avoid inherent challenges in performing higher order interpolation, typical of embedding techniques, by discretizing operators directly on the manifold. The proposed method is simple and requires the solution of a much smaller linear system, compared to the orthogonal gradient method.

2 The Radial Basis Function

Given function data $\{f_k\}_{k=1}^N$ at the node locations $\{x_k\}_{k=1}^N \in \Re^d$, the RBF interpolant $s(x)$ to the data is given as

$$s(x) = \sum_{i=1}^{N} \lambda_i \phi(\| x - x_i \|), \tag{1}$$

where $\phi(\| x - x_i \|)$ is the RBF kernel centered at the node x_i, and λ_i are coefficients chosen to satisfy the interpolation conditions

$$s(x_i) = f(x_i) \quad i = 1 \cdots N, \tag{2}$$

which is equivalent to solving the linear system

$$A\vec{\lambda} = \vec{f}. \tag{3}$$