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Using Analytic Solution Methods on Unsaturated Seepage Flow Computations

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Abstract
This paper describes a change of variables applied to Richards' equation for steady-state unsaturated seepage flow that makes the numerical representation of the new version of this highly nonlinear partial differential equation (PDE) much easier to solve, and the solution is significantly more accurate. The method is applied to two-dimensional unsaturated steady-state flow in a block of soil that is initially very dry until water is applied at the top. Both a quasi-linear version of relative hydraulic conductivity for which an analytic solution exists and a van Genuchten version of relative hydraulic conductivity are numerically solved using the original and new versions of the governing PDE. Finally, results of this research will be presented in this paper. It was found that for the test problem, the change-of-variables version of the governing PDE was significantly easier to solve and resulted in more accurate solutions than the original version of the PDE.

Keywords: Richards' equation, analytic methods, numerical solution of nonlinear partial differential equations

1 Introduction

Richards' equation (Richards, 1931) is a highly nonlinear equation that governs unsaturated seepage flow in soils. Various linearization schemes such as Newton and Picard methods (Putti and Paniconi, 1992 and Mehl, 2006), often used in conjunction with line search techniques (Tracy, 2009), have been applied to the numerical representation of this nonlinear partial differential equation (PDE) to iterate to a solution with varying degrees of success. Often, a comprehensive solution remains elusive. This challenge is especially difficult for steady-state problems. For transient solutions, the size of the time step can be easily decreased, thus giving the numerical solution more stability. There is no time step size in the steady-state solution. Some researchers have used a pseudo-transient approach (Tracy, et al., 2005) in which the time step is gradually increased to the point where a steady-state solution results. From the author's experience, this proved to be a delicate hit-and-miss solution for the most difficult problems.

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This research uses analytic methods that allow more accurate numerical solutions of the steady-state version of Richards' equation to be obtained using fewer nonlinear iterations. Analytic methods have been applied to groundwater modeling (Tracy, 2006 and 2007) in which either analytic solutions exist or techniques such as analytic element methods (Haitjema, 2005) are applied. A change of variables is used in the steady-state version of Richards' equation to improve accuracy and convergence of the numerical solution. Details will now be presented.

2 Governing Equations

2.1 Richards' Equation

Richards' equation for unsaturated steady-state seepage flow in a homogeneous, isotropic medium is given by

$$\frac{\partial}{\partial x}\left(k_r \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_r \frac{\partial h}{\partial y}\right) + \frac{\partial k_r}{\partial y} = 0 \tag{1}$$

where

h = pressure head (L)

k_r = relative hydraulic conductivity ($0 \leq k_r \leq 1$)

x = x coordinate (L)

y = y coordinate (L)

Eq. 1 is highly nonlinear because k_r is a function of h.

2.2 Relative Hydraulic Conductivity

There are many ways to compute relative hydraulic conductivity. These are typically rooted in data obtained from laboratory experiments or analyses of field data. Two methods will be discussed in this paper: the van Genuchten approximation (van Genuchten, 1980) and quasi-linear approximation (Warrick, 2003).

For $h < 0$, the van Genuchten approximation is given by

$$k_r = \frac{\{1 - (-\beta h)^{n-1}[1 + (-\beta h)^n]^{-m}\}^2}{[1 + (-\beta h)^n]^{m/2}} \tag{2}$$

where

n = modeling parameter that varies with soil type ($n > 0$)

$m = 1 - \frac{1}{n}$

β = modeling parameter that varies with soil type (L^{-1}) ($\beta > 0$)