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A Family of Multipoint Flux Mixed Finite Element Methods for Elliptic Problems on General Grids

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Abstract

In this paper, we discuss a family of multipoint flux mixed finite element (MFMFE) methods on simplicial, quadrilateral, hexahedral, and triangular-prismatic grids. The MFMFE methods are locally conservative with continuous normal fluxes, since they are developed within a variational framework as mixed finite element methods with special approximating spaces and quadrature rules. The latter allows for local flux elimination giving a cell-centered system for the scalar variable. We study two versions of the method: with a symmetric quadrature rule on smooth grids and a non-symmetric quadrature rule on rough grids. Theoretical and numerical results demonstrate first order convergence for problems with full-tensor coefficients. Second order superconvergence is observed on smooth grids.

Keywords: mixed finite element, multipoint flux approximation, cell-centered finite difference, full tensor, simplices, quadrilaterals, hexahedra, triangular prisms.

1. Introduction

We discuss the development of a family of numerical schemes for second order elliptic problems. These methods, referred to as multipoint flux mixed finite element (MFMFE) methods, allow for an accurate and efficient treatment of full tensor coefficients, irregular geometries and heterogeneities that require highly distorted grids and discontinuous coefficients. These schemes are shown to be cell-centered discretizations and to have convergent approximations for both the scalar variable and its flux. Following the terminology for Darcy flow, we refer to the scalar variable as pressure and the flux variable as velocity.

MFMFE methods can be viewed as variational counterpart of the multipoint flux approximation (MPFA) methods [1, 2, 3, 4]. In the MPFA finite volume framework, sub-edge (sub-face) fluxes are introduced, which allows for localization of velocity interactions around mesh vertices. Therefore fluxes can be easily eliminated, resulting in a

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cell-centered pressure scheme. Similar elimination is achieved in the MFMFE variational framework, by employing appropriate finite element spaces and special quadrature rules. Our approach is based on the BDM_1 [5] on triangles and quadrilaterals, the $BDDF_1$ [6] on tetrahedra or hexahedra, or the CD_1 [7] spaces on triangular prisms with a trapezoidal quadrature rule applied on the reference element. Related approaches have been developed in [8] on simplicial grids and in [9, 10] on quadrilateral grids using a broken Raviart-Thomas space. A key element of our methods is that the velocity space has n normal degrees of freedom on each edge (face), where n is the number of vertices of the edge (face). In the case of a reference element with some square faces, the original $BDDF_1$ (cube) and CD_1 (triangular prism) have only three degrees of freedom per face, which is insufficient for local flux elimination. We enhance the spaces by adding an appropriate number of curl basis functions, so that the resulting spaces have four degrees of freedom on square faces.

Due to their variational formulation, the MFMFE methods allow for multiscale and multiphysics extensions such as the mortar mixed finite element methods [11, 12, 13] and the enhanced velocity method [14]. These methods can handle non-matching grids and allow for coupling of different numerical algorithms and different physics in adjacent subdomains. The multiblock variational framework is useful in designing optimal parallel solvers that utilize efficient interface multiscale bases as interface preconditioners and subdomain solvers such as algebraic multigrid. These approaches have also been shown to be convergent and efficient when applying stochastic methods for uncertainty analyses [15, 16] and applying the MFMFE methods for multiscale modeling of nonlinear flow problems in porous media [17].

The remainder of the paper is organized as follows. The MFMFE methods on various grids are defined in Section 2. Theoretical convergence results are presented in Section 3. Numerical results confirming the theory are discussed in Section 4. Conclusions are given in Section 5.

2. Formulation of multipoint flux mixed finite element methods

Consider a boundary value problem on a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary $\partial\Omega$,

$$-\nabla \cdot \mathbb{A}\nabla p = f \quad \text{in } \Omega, \tag{2.1}$$

$$p = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

where p is an unknown scalar function, $\mathbb{A}$ is a symmetric, uniformly positive definite tensor with $L^\infty(\Omega)$ components, and f is a source term. The choice of homogeneous Dirichlet boundary conditions is made for simplicity of the presentation; other boundary conditions can also be treated. Let $H(\mathbf{div};\Omega) := \left\{\mathbf{v} \in (L^2(\Omega))^d : \nabla\cdot\mathbf{v} \in L^2(\Omega)\right\}$ and let $(\cdot,\cdot)$ denote the inner product in $L^2(\Omega)$. The weak mixed formulation of (2.1)-(2.2) reads: find $\mathbf{u} := -\mathbb{A}\nabla p \in H(\mathbf{div};\Omega)$ and $p \in L^2(\Omega)$, such that

$$(\mathbb{A}^{-1}\mathbf{u},\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = 0, \qquad \forall \mathbf{v} \in H(\mathbf{div};\Omega), \tag{2.3}$$

$$(\nabla\cdot\mathbf{u}, w) = (f, w), \qquad \forall w \in L^2(\Omega). \tag{2.4}$$

MFMFE methods have been developed and analyzed in [18] on simplicial and quadrilateral grids, in [19, 20, 13] on hexahedral grids and in [21] on triangular-prismatic grids. The method is defined as: find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in W_h$ such that

$$(\mathbb{A}^{-1}\mathbf{u}_h,\mathbf{v})_Q - (p_h, \nabla\cdot\mathbf{v}) = 0, \qquad \forall \mathbf{v} \in \mathbf{V}_h, \tag{2.5}$$

$$(\nabla\cdot\mathbf{u}_h, w) = (f, w), \qquad \forall w \in W_h \tag{2.6}$$

There are two key ingredients in the method. The first one is an appropriate choice of mixed finite element spaces $\mathbf{V}_h$ and W_h and degrees of freedom. The second one is a specific choice of the numerical integration rules for $(\cdot,\cdot)_Q$ in (2.5). These two choices allow for flux variables associated with a vertex to be expressed by cell-centered pressures surrounding the vertex. This results in a 9 point or 27 point pressure stencil on logically rectangular 2D or 3D grids.

The quadrature rule (2.22) can be symmetric or non-symmetric. We call the method symmetric or non-symmetric MFMFE method depending on the choice of quadrature rule. On affine or smooth grids, both the symmetric and non-symmetric MFMFE methods give first-order accurate velocities and pressures, as well as second order accurate

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