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A generic type of frequency-domain spectral element model for the dynamics of a laminated composite plate

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ABSTRACT

Various solution techniques have been developed in the last decades for accurate prediction of the dynamic responses of a laminated composite structure. The spectral element method (SEM) is well known as an exact solution method that provides extremely accurate dynamic responses even in the high-frequency region. In this study, we develop a spectral element model for a rectangular finite composite plate element. The present spectral element model is developed by modifying the boundary splitting method introduced in the previous studies of the authors. As a result, the four corner nodes of the rectangular finite composite plate element, which were inactive (fixed) in the previous studies, become active. Thus, the present spectral element model can be used as a generic type of finite element, which can be applied to any laminated composite plates with arbitrary boundary conditions. The accuracy and efficiency of the present spectral element model are evaluated by comparing its results with exact solutions and solutions using the commercial finite element analysis package, ANSYS. In addition, the vibration and wave characteristics are numerically investigated by varying the lay-ups of some examples of laminated composite plates with various geometries and boundary conditions.

1. Introduction

The structural properties of a laminated composite plate (simply, composite plate) can be controlled to satisfy the design requirements by modifying the fiber angles and the number of lamina. As composite plates with high specific strength and specific stiffness have been widely applied to many engineering fields including mechanical, aerospace, and civil engineering, it is very important to accurately predict the vibration characteristics of a composite plate during the design phase [1]. However, it is well known that exact vibration responses can be obtained only for very specific types of plates such as Levy-type plates, and various vibration analysis techniques that have been developed in the last decades can only obtain approximate solutions [2,3].

The finite element method (FEM) is one of the powerful computational methods that have been widely used to perform vibration analyses of complex structures. The accuracy of the FEM is closely related to the size of the finite elements used in the analysis. To obtain reliable FEM solutions, particularly in the high-frequency region, a structure must be discretized into many smaller finite elements, which will result in a significant increase of computational cost. Therefore, the frequency-domain spectral element method (SEM) can be considered as an alternative to the FEM.

Compared to the FEM, the SEM is known to provide exact solutions very efficiently by representing a uniform structure element as a single finite element regardless of its dimension [4]. This is true because the exact dynamic stiffness matrix (or spectral element matrix) formulated from the free wave solutions satisfying the governing equations in the frequency domain is used as the stiffness matrix in the analysis. Despite the outstanding features of the SEM, it has been applied mostly to one-dimensional structures [4,5].

There are very few SEM applications for two-dimensional (2D) structures. Langley [6], Berçin [7], and Leung and Zhou [8] presented the dynamic stiffness methods for the vibration analysis of the Levy-type isotropic thin plate, orthotropic thin plate, and laminated thick plate, respectively. Lee and Lee [9] applied the SEM to the Levy-type isotropic thin plate subjected to distributed dynamic loads. Hajheidari and Mirdamadi [10,11] applied the SEM to the vibration analysis of Levy-type symmetric and non-symmetric cross-ply composite plates. Orrenius [12] applied a semi-analytical FEM to isotropic, orthotropic, and rip-stiffened semi-infinite plates. Chakraborty and Gopalakrishnan [13,14] presented spectral element models for semi-bounded and semi-infinite plate elements. Casimir et al. [15] presented a dynamic stiffness matrix for isotropic plates with free edge boundary condi-

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tions. From the aforementioned literature, we find that the SEM applications to 2D structures have been limited to plates with very specific geometries and boundary conditions.

Birgersson et al. [16] and Mitra and Gopalakrishnan [17] presented spectral element models for rectangular plates by using a spectral super element method (SSEM). However their spectral element models can be assembled only in one plate direction (e.g., the x-direction). Park and his colleagues [18–20] developed spectral element models for membranes and isotropic and orthotropic composite plates by using a combination of the boundary splitting method [21] and SSEM [16]. Their spectral element models can be assembled in two plate directions (i.e., the x- and y-directions). However, as the four corner nodes of the spectral element models are inactive (fixed), their applications were limited to very specific types of plates. Recently, Park and Lee [22] developed a spectral element model for isotropic rectangular plates by modifying the boundary splitting method used in their previous study [19], in order to make the inactive four corner nodes active. Their spectral element model can be assembled in two plate directions without any limitation. Thus, it can be considered as a generic type of spectral element model that can be applied to any isotropic plate with arbitrary boundary conditions. However, to the authors' best knowledge, such a generic type of spectral element model has not been reported in the literature for composite plates with arbitrary boundary conditions.

Thus, the objective of this study is to develop a generic type of spectral element model that can be applied to any composite plates with arbitrary boundary conditions. The accuracy and performance of the newly developed spectral element model are then verified by comparing its results with those of exact solutions and solutions using the commercial finite element analysis package, ANSYS [23].

2. Governing differential equations of motion for a composite plate

2.1. Governing differential equations of motion

Consider a rectangular finite composite plate element made of m layers (or lamina) The xy-plane coincides with the mid-plane of the composite plate and the z-coordinate is normal to the mid-plane. The dimensions of the composite plate in the x- and y-directions are L_x and L_y, respectively, and the mass per unit area of the composite plate is ρ. The components of displacement at a point, occurring in the x, y, and z-directions are denoted by $u(x,y,t)$, $v(x,y,t)$, and $w(x,y,t)$, respectively. For the finite composite plate element subjected to the transverse force $f(x,y,t)$, the governing differential equations of motion are given by [1]:

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left[A_{11}\frac{\partial u}{\partial x}+A_{12}\frac{\partial u}{\partial y}+A_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-B_{11}\frac{\partial^2 w}{\partial x^2}-B_{12}\frac{\partial^2 w}{\partial y^2}-2B_{16}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+\frac{\partial}{\partial y}\left[A_{16}\frac{\partial u}{\partial x}+A_{26}\frac{\partial u}{\partial y}+A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-B_{16}\frac{\partial^2 w}{\partial x^2}-B_{26}\frac{\partial^2 w}{\partial y^2}-2B_{66}\frac{\partial^2 w}{\partial x\partial y}\right]=\rho\frac{\partial^2 u}{\partial t^2}\\
&\frac{\partial}{\partial x}\left[A_{16}\frac{\partial u}{\partial x}+A_{26}\frac{\partial v}{\partial y}+A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-B_{16}\frac{\partial^2 w}{\partial x^2}-B_{26}\frac{\partial^2 w}{\partial y^2}-2B_{66}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+\frac{\partial}{\partial y}\left[A_{12}\frac{\partial u}{\partial x}+A_{22}\frac{\partial v}{\partial y}+A_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-B_{12}\frac{\partial^2 w}{\partial x^2}-B_{22}\frac{\partial^2 w}{\partial y^2}-2B_{26}\frac{\partial^2 w}{\partial x\partial y}\right]=\rho\frac{\partial^2 v}{\partial t^2}\\
&\frac{\partial^2}{\partial x^2}\left[B_{11}\frac{\partial u}{\partial x}+B_{12}\frac{\partial v}{\partial y}+B_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2}-2D_{16}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+2\frac{\partial^2}{\partial x\partial y}\left[B_{16}\frac{\partial u}{\partial x}+B_{26}\frac{\partial v}{\partial y}+B_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-D_{16}\frac{\partial^2 w}{\partial x^2}-D_{26}\frac{\partial^2 w}{\partial y^2}-2D_{66}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+\frac{\partial^2}{\partial y^2}\left[B_{12}\frac{\partial u}{\partial x}+B_{22}\frac{\partial v}{\partial y}+B_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-D_{12}\frac{\partial^2 w}{\partial x^2}-D_{22}\frac{\partial^2 w}{\partial y^2}-2D_{26}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+f(x,y,t)=\rho\frac{\partial^2 w}{\partial t^2}
\end{aligned}
\tag{1}
$$

where A_{ij} are the extensional stiffnesses, B_{ij} are the bending-extensional coupling stiffnesses, and D_{ij} are the bending stiffnesses, which are defined in [1].

2.2. Governing differential equations in the frequency domain

The governing differential equations of motion, Eq. (1), can be transformed into the frequency domain by using the fast Fourier transform (FFT) theory. The displacement fields and external force can be represented in the spectral forms as follows [5]:

$$
\begin{aligned}
&\{u(x,y,t),\ v(x,y,t),w(x,y,t),f(x,y,t)\}\\
&\quad=\frac{1}{N}\sum_{n=0}^{N-1}\{\bar{u}_n(x,y),\ \bar{v}_n(x,y),\bar{w}_n(x,y),\bar{f}_n(x,y)\}e^{i\omega_n t}
\end{aligned}
\tag{2}
$$

where N is the number of samples for the FFT-based spectral analysis, $i=\sqrt{-1}$ is the imaginary unit, and $\omega_n = 2\pi n/T\ (n = 0,1,2,\ldots,N)$ are the discrete frequencies up to the Nyquist frequency. The overbarred quantities (i.e., $\bar{u}_n(x,y), \bar{v}_n(x,y), \bar{w}_n(x,y)$, and $\bar{f}_n(x,y)$) with subscripts n represent the spectral or Fourier components of the corresponding time domain quantities (i.e., $u(x,y,t)$, $v(x,y,t)$, $w(x,y,t)$, and $f(x,y,t)$). For the sake of brevity, the over-bars and subscripts, n, if not necessary, will be omitted in the following derivations.

By substituting Eq. (2) into Eq. (1), the governing differential equations of motion can be transformed into the frequency domain as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left[A_{11}\frac{\partial u}{\partial x}+A_{12}\frac{\partial u}{\partial y}+A_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-B_{11}\frac{\partial^2 w}{\partial x^2}-B_{12}\frac{\partial^2 w}{\partial y^2}-2B_{16}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+\frac{\partial}{\partial y}\left[A_{16}\frac{\partial u}{\partial x}+A_{26}\frac{\partial u}{\partial y}+A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-B_{16}\frac{\partial^2 w}{\partial x^2}-B_{26}\frac{\partial^2 w}{\partial y^2}-2B_{66}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+\rho\omega^2 u=0\\
&\frac{\partial}{\partial x}\left[A_{16}\frac{\partial u}{\partial x}+A_{26}\frac{\partial v}{\partial y}+A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-B_{16}\frac{\partial^2 w}{\partial x^2}-B_{26}\frac{\partial^2 w}{\partial y^2}-2B_{66}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+\frac{\partial}{\partial y}\left[A_{12}\frac{\partial u}{\partial x}+A_{22}\frac{\partial v}{\partial y}+A_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-B_{12}\frac{\partial^2 w}{\partial x^2}-B_{22}\frac{\partial^2 w}{\partial y^2}-2B_{26}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+\rho\omega^2 v=0\,\frac{\partial^2}{\partial x^2}\left[B_{11}\frac{\partial u}{\partial x}+B_{12}\frac{\partial v}{\partial y}+B_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2}-2D_{16}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+2\frac{\partial^2}{\partial x\partial y}\left[B_{16}\frac{\partial u}{\partial x}+B_{26}\frac{\partial v}{\partial y}+B_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-D_{16}\frac{\partial^2 w}{\partial x^2}-D_{26}\frac{\partial^2 w}{\partial y^2}-2D_{66}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+\frac{\partial^2}{\partial y^2}\left[B_{12}\frac{\partial u}{\partial x}+B_{22}\frac{\partial v}{\partial y}+B_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-D_{12}\frac{\partial^2 w}{\partial x^2}-D_{22}\frac{\partial^2 w}{\partial y^2}-2D_{26}\frac{\partial^2 w}{\partial x\partial y}\right]\\
&\quad+\rho\omega^2 w+f(x,y;\omega)=0
\end{aligned}
\tag{3}
$$

3. Weak form of governing differential equations

In this study, we formulate the spectral element model for a finite composite plate element by using a variational method [5]. To that end, the weak form of the governing differential equations in the frequency domain given by Eq. (3) can be obtained in the following form:

$$
\begin{aligned}
&\iint_A\left[\frac{\partial\delta u}{\partial x}\left\{A_{11}\frac{\partial u}{\partial x}+A_{12}\frac{\partial v}{\partial y}+A_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-B_{11}\frac{\partial^2 w}{\partial x^2}-B_{12}\frac{\partial^2 w}{\partial y^2}-2B_{16}\frac{\partial^2 w}{\partial x\partial y}\right\}\right.\\
&-\frac{\partial^2\delta w}{\partial x^2}\left\{B_{11}\frac{\partial u}{\partial x}+B_{12}\frac{\partial v}{\partial y}+B_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2}-2D_{16}\frac{\partial^2 w}{\partial x\partial y}\right\}\\
&+\frac{\partial\delta v}{\partial y}\left\{A_{12}\frac{\partial u}{\partial x}+A_{22}\frac{\partial v}{\partial y}+A_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-B_{12}\frac{\partial^2 w}{\partial x^2}-B_{22}\frac{\partial^2 w}{\partial y^2}-2B_{26}\frac{\partial^2 w}{\partial x\partial y}\right\}\\
&-\frac{\partial^2\delta w}{\partial y^2}\left\{B_{12}\frac{\partial u}{\partial x}+B_{22}\frac{\partial v}{\partial y}+B_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-D_{12}\frac{\partial^2 w}{\partial x^2}-D_{22}\frac{\partial^2 w}{\partial y^2}-2D_{26}\frac{\partial^2 w}{\partial x\partial y}\right\}\\
&+\left(\frac{\partial\delta u}{\partial y}+\frac{\partial\delta v}{\partial x}\right)\left\{A_{16}\frac{\partial u}{\partial x}+A_{26}\frac{\partial v}{\partial y}+A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-B_{16}\frac{\partial^2 w}{\partial x^2}-B_{26}\frac{\partial^2 w}{\partial y^2}-2B_{66}\frac{\partial^2 w}{\partial x\partial y}\right\}\\
&-2\frac{\partial^2\delta w}{\partial x\partial y}\left\{B_{16}\frac{\partial u}{\partial x}+B_{26}\frac{\partial v}{\partial y}+B_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-D_{16}\frac{\partial^2 w}{\partial x^2}-D_{26}\frac{\partial^2 w}{\partial y^2}-2D_{66}\frac{\partial^2 w}{\partial x\partial y}\right\}\\
&\left.+\rho\omega^2(\delta uu+\delta vv+\delta ww)\right]dA=\iint_A f(x,y)\delta w dA+\int_y N_{x1}\delta u(-\tfrac{1}{2}L_x,y)dy\\
&+\int_y N_{x2}\delta u(\tfrac{1}{2}L_x,y)dy+\int_x N_{xy1}\delta u(x,-\tfrac{1}{2}L_y)dx+\int_x N_{xy2}\delta u(x,\tfrac{1}{2}L_y)dx\\
&+\int_y N_{xy3}\delta v(-\tfrac{1}{2}L_x,y)dy+\int_y N_{xy4}\delta v(\tfrac{1}{2}L_x,y)dy+\int_x N_{y1}\delta v(x,-\tfrac{1}{2}L_y)dx\\
&+\int_x N_{y2}\delta v(x,\tfrac{1}{2}L_y)dx+\int_y V_{x1}\delta w(-\tfrac{1}{2}L_x,y)dy+\int_y V_{x2}\delta w(\tfrac{1}{2}L_x,y)dy\\
&+\int_x V_{y1}\delta w(x,-\tfrac{1}{2}L_y)dx+\int_x V_{y2}\delta w(x,\tfrac{1}{2}L_y)dx+\int_y M_{x1}\delta(\partial w(-\tfrac{1}{2}L_x,y)/\partial x)dy\\
&+\int_y M_{x2}\delta(\partial w(\tfrac{1}{2}L_x,y)/\partial x)dy+\int_x M_{y1}\delta(\partial w(x,-\tfrac{1}{2}L_y)/\partial y)dx\\
&+\int_x M_{y2}\delta(\partial w(x,\tfrac{1}{2}L_y)/\partial y)dx
\end{aligned}
\tag{4}
$$

where $V_{x1}(y)$, $V_{x2}(y)$, $V_{y1}(x)$, and $V_{y2}(x)$ are the external transverse shear forces acting on the four edges of a finite composite plate ele-