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An analytical framework to extend the general structural stability analysis by considering certain inelastic effects—theory and application to delaminated composites

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ABSTRACT

An analytical framework which incorporates damage propagation/growth into the general structural stability analysis is presented. Therefore, the conventional total potential energy approach is extended by introducing an *extended total potential energy-like functional* capable of describing inelastic processes in which equilibrium holds between available and the required force for producing a change in structure. The work deals with systems which are described by *I* generalized coordinates and *K* damage parameters. The damage parameters are found to be functions of *I* generalized coordinates and *M* load parameters. The underlying variational principle for inelastic solids may be solved using discrete formulations or approximate methods such as a RAYLEIGH–RITZ formulation. This leads to a set of non-linear algebraic equations, comprising post-critical equilibrium paths and damage propagation. In order to verify the framework, it is applied to the well-known problem in which a delaminated composite strut/plate is subjected to an in-plane compressive load.

1. Introduction

Structural stability problems are often analyzed using the total potential energy (TPE) approach, as documented in established textbooks such as [1,30]. The description of a system's response by a set of generalized coordinates (GCs) q_i often allows the modelling of various stability phenomena by means of just a few degrees of freedom; for instance, see [13,31] regarding stability phenomena in sandwich structures and [7,32] for plated structures. However, applying the structural stability analysis framework, as described in [1,30], exhibits one major limitation—its restriction to conservative processes. Hence, changes in structure, *e.g.* damage propagation, may not be considered. However, stability phenomena are often linked to material damage or failure. This also counts for a main area of the structural stability analysis which is the subject of interest of the current work—the buckling and postbuckling behaviour of slender structures such as struts, plates and shells. Moreover, application to composite materials is also sought after since such materials are increasingly used nowadays, specifically once lightweight constructions are desired. Hence, damage mechanics relevant to composites are required to be considered when such structures are subjected to compressive loads. Such phenomena include for instance, matrix cracking, fibre failure, fibre kinking, delamination initiation and delamination growth, *e.g.* see [14,17,18,28,33].

A well-known example which requires both structural stability analysis and damage mechanics is the (post-)buckling response of composite plates containing an initial delamination. This problem has been studied by many researchers since the 1980s, starting with the work from Chai et al. [3,4]. However, all pre-existing (semi-)analytical models are limited to stationary delaminations, *i.e.* non-growing delaminations. Postbuckling solutions are provided for stationary delaminations, and damage analysis (*e.g.* examining the strain energy release rate) reveals if growth occurs and whether such growth is stable or unstable. However, post-critical responses beyond the state where delamination growth occurs are yet solely modelled using finite element (FE) simulations (*e.g.* see [20]). In the current work, with the aid of the framework proposed, structural stability analysis and damage propagation are combined into one single formulation. Thus, post-critical responses beyond the state where delamination growth occurs can be determined.

Conditions for (in-)stability for inelastic solids based on variational principles and energy criteria are well documented within the literature. Starting with [10,11], it is shown that equilibrium

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equations and their stability are determined by a variational principle derived from the underlying general boundary value problem. Extensions to these studies are made in [22,23], allowing for a more general formulation and application to deformation paths (relevant addition to equilibrium solutions if path-dependent solids are subject of interest).

In the work presented, such a variational principle as well as a method to derive strain energy-like potentials [19,26,27] are applied to the structural stability analysis formalism. Active damage parameters are described as functions of I generalized coordinates and M load parameters. Hence, the total work of deformation (consisting of the strain energy and energy associated with K inelastic effects) may be described by a potential which allows the derivation of an *extended total potential energy* (ETPE) functional. By describing the system using a discrete formulation or implementing an approximate method, *e.g.* a RAYLEIGH–RITZ formulation, the solution of the underlying variational principle yields the deformation paths for the irreversible process. With the solution obtained, the evolution of the damage parameters is then readily determined. Moreover, the stability of the equilibrium solutions may be evaluated using the governing functional. The outcome of the current work is an analytical framework which enables the modelling of stability related phenomena of structures by means of just a few GCs. Thus, efficient simulations of pre-damaged slender structures, comprising post-critical deformation paths and possible damage propagation, are enabled. This depicts an extension to the conventional structural stability analysis.

2. Analytical framework

2.1. Total potential energy principle and its limitation

The total potential energy (Π) of a structural system described by I generalized coordinates (q_i) is defined as [1,30],

$$\Pi(q_i, A_m) = W(q_i) - A_m\alpha_m(q_i), \tag{1}$$

with $i = 1, 2, \ldots, I, m = 1, 2, .., M$ and W being the strain energy. The conjugate variables prescribed loads (A_m) and their respective displacements (α_m) describe the work done by external forces. If not explicitly stated otherwise, the summation convention is employed in which a repeated index is to be summed over its entire range. The prescribed loads A_m and the conjugate displacements α_m may also be understood as independent generalized forces and generalized displacements respectively. Applying the well-known variational principle, $\delta\Pi(q_i) = 0$, yields the equilibrium solutions, *i.e.* the deformation paths, $A_m(q_i)$.

In Eq. (1), the current state of damage may be considered by a set of damage parameters ξ_k. However, the assumption that all damage parameters do not change must hold (reversible processes). Thus, the total potential energy (TPE) given in Eq. (1) may be expressed as

$$\Pi(q_i, \xi_k, A_m) = W(q_i, \xi_k) - A_m\alpha_m(q_i, \xi_k), \quad \text{for} \quad \xi_k = \text{const.}, \tag{2}$$

in which the dependency in regards to the current state of damage is accounted for (with $k = 1, 2, \ldots, K$). Thus, only the deformation behaviour under a given constant state of damage may be modelled which limits the applicability of the TPE principle.

2.2. Extended total potential energy principle

An analytical framework which enables the modelling of the (post-)buckling responses by a set of GCs (q_i), without the limitation of a non-evolving state of damage (ξ_k are not assumed constant anymore), requires the derivation of an extended total potential energy-like functional, $\Pi^*(q_i)$. In order to derive such a functional, the method of deriving strain energy-like potentials [27] is applied to systems described by I GCs. Subsequently, the variational principle for inelastic solids (see *e.g.* [10,11,22]) may be applied yielding the deformation path for an evolving state of damage.

2.2.1. Potential character of the total work of deformation and the extended total potential energy

The sufficient condition for the total work of deformation being a potential is derived next. Therefore, in the spirit of [27], the response of the system is described in terms of generalized displacements (α_m) and generalized forces (A_m). Consequently, it is necessary to differentiate between prescribed displacements (displacement controlled setups) and prescribed loads (load controlled setups). For prescribed displacements, the governing functional is the strain energy which is a function (W_{rev}) of the independent generalized displacements (α_m) and the current state of damage defined by K damage parameters (ξ_k), *i.e.* $W_{\text{rev}} = W_{\text{rev}}(\alpha_m, \xi_k)$. On the other hand, once prescribed loads are given, the problem is described by the total potential energy which is a function of the independent generalized forces (A_m) and the damage parameters (ξ_k), *i.e.* $V = V(A_m, \xi_k)$.

2.2.1.1. Prescribed displacements. In order to describe the total work of deformation W_{tot} as a potential, *i.e.* $A_m = \partial W_{\text{tot}}/\partial\alpha_m$ and consequently $W_{\text{tot}} = \int A_m \mathrm{d}\alpha_m$, the strain energy is examined for a change in structure (*e.g.* increase in damage). The total derivative of the strain energy yields

$$\mathrm{d}W_{\text{rev}} = \left.\frac{\partial W_{\text{rev}}}{\partial\alpha_m}\right|_{\xi_k} \mathrm{d}\alpha_m + \left.\frac{\partial W_{\text{rev}}}{\partial\xi_k}\right|_{\alpha_m} \mathrm{d}\xi_k \quad \text{with} \quad f_k = -\frac{\partial W_{\text{rev}}}{\partial\xi_k}, \tag{3}$$

where f_k is the thermodynamic force [25] associated to the kth damage parameter, *i.e.* the force available for producing a change in the structure [27]. The total work is equal to the sum of elastic energy (W_{rev}) plus energy associated with dissipation (W_{d}), thus

$$W_{\text{tot}}(\alpha_m, \xi_k) = W_{\text{rev}}(\alpha_m, \xi_k) + W_{\text{d}}(\xi_k). \tag{4}$$

The dissipative energy is assumed to be a state function depending on the damage parameters only [27]. The force required for a change in structure (g_k), *i.e.* the change in dissipative energy with respect to the damage parameter (which may be a material parameter), reads

$$g_k = \frac{\partial W_{\text{d}}}{\partial\xi_k}. \tag{5}$$

If the force available for a change in structure (f_k) is equal to the force required (g_k), Eqs. $(3)_2$ and (5) yield

$$f_k(\alpha_m, \xi_k) = g_k(\xi_k) \Rightarrow -\frac{\partial W_{\text{rev}}}{\partial\xi_k} = \frac{\partial W_{\text{d}}}{\partial\xi_k}. \tag{6}$$

Eq. (6) outlines the sufficient condition to describe the total work of deformation as a potential. From the set of equations in Eq. $(6)_2$ the damage parameters ξ_k may be derived in terms of independent generalized displacements α_m, *i.e.* $\xi_k = \xi_k(\alpha_m)$. Hence, Eq. $(6)_2$ may be also understood as the evolution law for the damage parameters ξ_k. It is assumed that Eq. $(6)_2$ provides a unique solution for the damage parameters ξ_k, differentiable in α_m [27]. By inserting $\xi_k(\alpha_m)$ in Eq. (4), W_{tot} is found as a function of the independent generalized displacements, $W_{\text{tot}} = W_{\text{tot}}(\alpha_m)$, which describes a potential of the generalized forces. It should be noted that Eqs. $(3)_2$ and (6) apply for systems where the boundary is subjected to prescribed displacements.

2.2.1.2. Prescribed loads. For load controlled setups (generalized forces serve as independent variables), the thermodynamic force is equal to the change of the total potential energy, $V = V(A_m, \xi_k)$,