Computers and Geotechnics 82 (2017) 16–30

Contents lists available at ScienceDirect

Computers and Geotechnics

journal homepage: www.elsevier.com/locate/compgeo

Research Paper

Application of fractional calculus in modelling ballast deformation under cyclic loading

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ARTICLE INFO

Article history:
Received 20 May 2016
Received in revised form 19 September 2016
Accepted 26 September 2016

Keywords:
Ballast
Constitutive relations
Cyclic loading
Fractional calculus

ABSTRACT

Most constitutive models can only simulate cumulative deformation after a limited number of cycles. However, railroad ballast usually experiences a large number of train passages that cause history-dependent long-term deformation. Fractional calculus is an efficient tool for modelling this phenomenon and therefore is incorporated into a constitutive model for predicting the cumulative deformation. The proposed model is further validated by comparing the model predictions with a series of corresponding experimental results. It is observed that the proposed model can realistically simulate the cumulative deformation of ballast from the onset of loading up to a large number of load cycles.

1. Introduction

Ballast usually serves as an essential track construction layer to bear the load transmitted by railroad ties, and also to facilitate rapid drainage. During the whole operation period, a rail track usually experiences a large number of train passages that cause a cumulative deformation of the underlying ballast. Accurate prediction of the corresponding maintenance periods necessitates the development of an advanced constitutive model that captures ballast deformation and degradation. Although traditional elastoplastic constitutive models have been investigated widely and successfully applied in many fields, more effort is required to realistically describe the stress-strain relationship of ballast subjected to long term cyclic loads. Traditional plasticity approaches, including elastoplastic models [1,2], generalised plasticity models [3,4], and bounding surface plasticity models [5–9], are capable of incorporating cyclic loading, but often consider very limited cycles ($N < 100$). For predicting cumulative deformation under a large number of cycles ($N \geqslant 10^3$), these models can be inaccurate due to the inevitable accumulation of numerical errors associated with finite element analysis. There is little possibility of using these theoretical models in practical engineering where the loading usually consists of at least tens of thousands of cycles.

To overcome the above limitations, various empirical and semi-empirical models have been proposed. Although empirical models are usually problem-targeted and easy to use in engineering applications [10,11], they do not reflect the essential mechanisms explaining aggregates degradation and related deformation. Semi-empirical models usually provide an alternative way to model cumulative deformation. For example, Suiker and de Borst [12] proposed an elasto-plastic methodology for simulating the cyclic deterioration of rail tracks by assuming that permanent deformation was caused by frictional sliding and volumetric compaction. The growth of each component of deformation was empirically simulated by a power law. François et al. [13] proposed an explicit elastoplastic model by assuming an exponential decrease of the accumulated strain. Indraratna et al. [14] proposed a pressure-dependent elasto-plastic model by introducing empirical parameters to consider the effect of particle breakage, stress ratio and number of load cycles. However, some parameters in these models require extensive and special laboratory tests and as such are often unattractive in railway problems.

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http://dx.doi.org/10.1016/j.compgeo.2016.09.010
0266-352X/

List of notations

A	is the constant of proportionality
$\mathbf{C}^e$	is the elastic compliance matrix
D	means derivation
E	is the material constant
G	is the shear modulus
H	is the plastic modulus
I	means integral
K	is the bulk modulus
L	is the particle diameter
M	is the critical state friction parameter
N	is the number of load cycle
N_s	is the number of particles
R_d	is the relative density
b	is the fitting parameter
e_0	is the initial void ratio
f	is the load frequency
k	is the degradation rate of minimum sized particles
l_s	is the particle size
$\mathbf{m}$	is the plastic flow tensor
$\mathbf{n}$	is the loading direction tensor
p'	is the mean effective principal stress
p'_0	is the initial mean effective principal stress
$\bar{p}'$	is the mean effective principal stress on bounding surface
$\bar{p}'_0$	is the initial mean effective principal stress on bounding surface
p_r	is the unit pressure
q	is the deviator stress
$\bar{q}$	is the deviator stress on bounding surface
R	is the ratio between energy dissipation by particle rearrangement and breakage energy
α	is the fractional order
β	is the fitting parameter
δ_{in}	is the distance from the stress origin to the image stress point
δ	is the distance from the current stress point to the image stress point
ρ	is the scalar relating the image stress and the loading stress
η	is the stress ratio between deviator stress and mean effective principal stress
χ	is the plastic multiplier
λ	is the gradient of the critical state line
κ	is the gradient of the swell line
ν	is the Poisson ratio
γ	is the plastic flow parameter
γ_s	is the surface shape factor
$\Gamma(\bullet)$	is the gamma function
Δ_s	is the fractal dimension of the aggregates
$\dot{\boldsymbol{\sigma}}$	is the incremental stress tensor
σ'_1	is the major effective principal stress
σ'_2	is the medium effective principal stress
σ'_3	is the minor effective principal stress
$\dot{\varepsilon}$	is the incremental strain tensor
$\dot{\varepsilon}^e$	is the incremental elastic strain tensor
$\dot{\varepsilon}^p$	is the incremental plastic strain tensor
$\dot{\varepsilon}_1^p$	is the major plastic principal strain
$\dot{\varepsilon}_3^p$	is the minor plastic principal strain
ε_v^e	is the elastic volumetric strain
ε_s^e	is the generalised elastic shear strain
$\dot{\varepsilon}_v^p$	is the plastic volumetric strain
$\dot{\varepsilon}_s^p$	is the generalised plastic shear strain
ϕ_{cr}	is the critical state friction angle
$\dot{S}_s$	is the incremental surface area of aggregates
Ω	is the surface energy

Rather than selecting different modelling techniques to describe experimentally observed stress-strain behaviour, a fundamental question that arises is: are we using the correct mathematical tools to describe material deformation? More precisely, in view of the topic of this paper, one may ask: are commonly used increments in a particular model correctly assumed as an integer order or should one choose more general operators of a fractional order? The answer to such a question is not obvious. In fact, the cumulative deformation of granular soils under cyclic loading is not only influenced by the current loading stress but also by previous loading cycles [15]. It is indeed a memory-intensive phenomenon that can be represented mathematically by using the concept of fractional calculus.

Koeller [16] developed fractional calculus for the theory of viscoelasticity to form a link between the ideal solid state, governed by Hooke's law, and the ideal liquid state, governed by Newton's law of viscosity. In the limit of an ideal solid the system has perfect memory while in the ideal liquid state it has no memory. Hence intermediate states, representing real materials, have imperfect memory and require fractional calculus to be modelled appropriately. Fractional calculus has been used successfully for problems involving soil mechanics, solid mechanics, vibration and damping.

Geotechnical applications using fractional calculus include the creep and relaxation behaviour of composite soil [17], the time dependence of Poisson's ratio [18], the strain hardening and softening behaviour of sand and clay under monotonic loading [19], the vibration of rail pads [20], and the anomalous diffusion of underground water [21,22].

However, for the problem of interest here, the cumulative deformation of ballast subjected to a large number of load cycles, further modelling and investigation is still required. This paper aims to develop a more rigorous model for predicting the cumulative deformation of ballast subjected to a large number of loading cycles. Traditional elastoplasticity theory is used and modified by incorporating the concept of fractional calculus, and then the developed model is validated against the results of a series of laboratory test results.

2. Fractional calculus

In traditional calculus the nth derivative or integral of a function is defined for n taking integer values only. In fractional calculus the definitions of a derivative and an integral are generalised and n can be a non-integer. One method to generalise the definition of the repeated integral, which results in the Riemann-Liouville fractional integral of the function, $z(x)$, is given by [23]:

$$ {}_0I_x^{\alpha} z(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{z(\tau)\mathrm{d}\tau}{(x-\tau)^{1-\alpha}}, \quad x > 0 \tag{1} $$

where I signifies an integral. α is the fractional order, ranging from 0 to 1, which can be correlated to the fractal dimension of a given granular material, as indicated in Appendix A. x denotes the independent variable. In this context, x can be regarded as the loading time in a static load test or the number of loading cycles in a cyclic load test. The conventional gamma function $\Gamma(x)$ is defined as: