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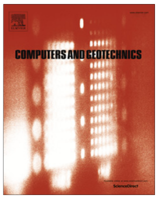

Research Paper

# Coupled consolidation in unsaturated soils: An alternative approach to deriving the Governing Equations

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ABSTRACT

The equations governing coupled consolidation in unsaturated soils are known to contain additional parameters when compared to the equations for saturated soils. Nonetheless, the variation of these parameters with suction or degree of saturation is not generally agreed upon. The paper introduces a novel approach to deriving general equations for each of these parameters and their variation, and explains that, for consistency with the constitutive and soil-water retention curve models adopted, these general equations need to be transformed into case-specific expressions. Finally, a conceptual model is presented highlighting how the behaviour of unsaturated soil reflects aspects of its water content.

## 1. Introduction

Several attempts have been made to model hydro-mechanical coupling in unsaturated soil states, usually based on extending Biot's theory [2]. Different simplifications can be made in extending Biot's approach, such as assuming that the air phase is drained and air pressure is equal to atmospheric pressure (e.g. [15,33,22,23,31]). This assumption applies to the whole of the unsaturated region, independently of the water and air content of the soil, and the continuity equation for the air phase is ignored. The assumption of a static air phase may impede the applicability of the Governing Equations to a series of Geotechnical Engineering problems involving air pressure, such as Reservoir Engineering, where both water and gas exist under pressure. In certain cases the flow of air is explicitly modelled and air pressure is an additional primary variable (e.g. [4,19,20,9,21,13]), yet transition between fluid phases may be ignored. The effect of temperature can be included in thermo-hydro-mechanical coupling in unsaturated soils (e.g. [16,26]).

The work presented here is an extension of the work done by Darkshanamurthy et al. [4] and Wong et al. [33]. Although in Darkshanamurthy et al. [4] the flow of the gas phase is modelled, "the effect of air diffusing through water, air dissolving in the water phase and the movement of water vapour are ignored". Wong et al. [33] extended the work of Darkshanamurthy et al. [4] to multidimensional cases and presented the Finite Element (FE) formulation for coupled consolidation problems, assuming that "(i) the pore-air pressure is atmospheric and remains unchanged during an analysis, and (ii) water flows through the soil skeleton in accordance with Darcy's law". These two assumptions were also made in the proposed approach.

The original formulations of the governing equations in Biot [2], Darkshanamurthy et al. [4], Wong et al. [33] are explained in Appendix A and the main points are discussed in the subsequent section, in order to highlight the differences of the proposed approach. All the relevant equations reported from the literature have been reproduced in Appendix A employing the same symbols as in the original publication, with the exception of Poisson's ratio which is always $\mu$. These equations are summarised in Table 1.

The ground-breaking work of Biot [2] set the basis for coupled consolidation analysis in saturated soils and in soils containing air in the form of occluded bubbles (i.e. not in a continuous form). The two equations proposed, i.e. the constitutive relationship for the soil structure and the constitutive relationship for the water phase, contain four moduli: $E$, which is Young's modulus (the shear modulus $G$ is used for shear strains); $H$, which is similar to Young's modulus and shows the effect of changing pore water pressure on the direct strains in the soil; $H_1$, which is a physical constant describing the effect of changes in applied total stress on the water

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**Table 1**
Summary of constitutive relationships proposed for the soil structure and for the water phase.

| | Constitutive relat. for the soil structure | Constitutive relat. for the water phase |
|---|---|---|
| Biot [2] | $\varepsilon_x = \frac{\sigma_x}{E} - \frac{\mu}{E}(\sigma_y + \sigma_z) + \frac{\sigma}{3H}$ (A.1)<br>and<br>$\gamma_{xy} = \frac{\tau_{xy}}{G}$ (A.2) | $\theta = \frac{1}{3H_1}(\sigma_x + \sigma_y + \sigma_z) + \frac{\sigma}{R}$ (A.3)<br>or<br>$\theta = \alpha\varepsilon + \frac{\sigma}{Q}$ (A.4)<br>where<br>$\alpha = \frac{2(1+\mu)}{3(1-2\mu)} \cdot \frac{G}{H}$ (A.5)<br>$Q = \frac{1}{R} - \frac{\alpha}{H}$ (A.6) |
| | Asumption: $H = H_1$<br>Approach based on the use of moduli $G$, $R$ and $H$ | |
| Darkshanamurthy et al. [4] | $\varepsilon_x = \frac{(\sigma_x - u_a)}{E_1} - \frac{\mu}{E_1}(\sigma_y + \sigma_z - 2u_a) + \left(\frac{u_a - u_w}{H_1}\right)$ (A.10)<br>and<br>$\gamma_{xy} = \frac{\tau_{xy}}{G}$ (A.2) | $\theta_w = \frac{(\sigma_x + \sigma_y + \sigma_z - 3u_a)}{3H'_1} + \frac{u_a - u_w}{R_1}$ (A.11)<br>or<br>$\theta_w = \frac{\beta}{3}\varepsilon + \gamma(u_a - u_w)$ (A.12)<br>where<br>$\beta = \frac{E_1}{H_1}\frac{1}{1-2\mu}$ (A.13)<br>$\gamma = \frac{1}{R_1} - \frac{\beta}{H'_1}$ (A.14) |
| | Asumption: $H_1 = H'_1$<br>Approach based on the use of moduli $E_1$, $R_1$ and $H_1$ | |
| Wong et al. [33] | $\varepsilon_{ij} = \frac{1+\mu}{E}\sigma^n_{ij} - \frac{\mu}{E}\sigma^n\delta_{ij} + \frac{u_a - u_w}{H}\delta_{ij}$ (A.17) | $\theta_w = \beta\varepsilon_v + \omega(u_a - u_w)$ (A.19)<br>where<br>$\beta = \frac{E}{H} \cdot \frac{1}{1-2\mu}$ (A.20)<br>$\omega = \frac{1}{R} - \frac{3\beta}{\mathrm{H}}$ (A.21) |
| | Asumption: No distinction between modulus $H$ in Eqs. (A.17) and (A.20)<br>Approach based on the use of moduli $E$, $R$ and $H$ | |
| Proposed approach | $\varepsilon_x = \frac{(\sigma_x - u_a)}{E} - \frac{\mu}{E}(\sigma_y + \sigma_z - 2u_a) + \left(\frac{u_a - u_w}{H}\right)$ (1)<br>and<br>$\gamma_{xy} = \frac{\tau_{xy}}{G}$ (1) | $\theta_w = \frac{(\sigma_x + \sigma_y + \sigma_z - 3u_a)}{E_w} + \frac{(u_a - u_w)}{R}$ (2)<br>or<br>$\theta_w = \Omega\varepsilon_{vol} + \omega(u_a - u_w)$ (10)<br>where<br>$\Omega = \frac{E}{E_w}\frac{1}{(1-2\mu)}$ (9)<br>$\omega = \left(\frac{1}{R} - \frac{3\Omega}{H}\right)$ (11) |
| | Assumption: moduli $H$ and $E_w$ are independent<br>Approach based on the use of moduli $E$, $E_w$, $R$ and $H$ | |

content; and $R$, which is a physical constant describing the effect of incremental pore water pressure on the water content.

Biot [2] demonstrated that $H = H_1$ for the particular stress condition where $\sigma_x = \sigma_y = \sigma_z$ and $\tau_{xy} = \tau_{yz} = \tau_{xz} = 0$, i.e. no distinction is necessary between the modulus governing the effect of changing pore water pressure on the direct strains and the modulus governing the effect of incremental pore water pressure on the water content.

In extending Biot's work to unsaturated soil conditions, Darkshanamurthy et al. [4] and Wong et al. [33] also assumed, either explicitly or implicitly that the two moduli remain equal and, as in Biot [2], are physical constants. However, there is no particular reason why the modulus governing the effect of matric suction on the direct strains should be equal to the modulus governing the effect of net stress on the volumetric water content under all circumstances and especially for truly unsaturated soils containing continuous air, where the effect on soil behaviour of changing suction is distinctively and fundamentally different from the effect of changing applied stress [5,10]. As discussed in the subsequent section, not making a clear distinction between the moduli controlling the effect of suction on direct strains and the effect of net stresses on volumetric water content oversimplifies the complex behaviour of unsaturated soils.

This drawback is addressed in the present paper, where a distinction between the two moduli is explicitly made. In addition to moduli $E$ and $R$, which are similar to the moduli in Biot [2], modulus $E_w$, which governs the effect of net stress on the volumetric water content, and modulus $H$, which governs the effect of matric suction on direct strains, are required for the formulation of the Governing Equations in unsaturated states. Following an approach similar to Biot [2] and Wong et al. [33], the constitutive relationship for the water phase is rewritten in a form containing three parameters, $\Omega$, $\omega$ and $H$, which are related to the four moduli and which are required to extend coupled consolidation to unsaturated soil states.

The main differences of the current approach, which constitute the innovative aspects of this work, are:

(a) a clear distinction is made between the two moduli controlling the effect of matric suction on direct strains, $H$, and the effect of net stress on the volumetric water content, $E_w$, as explained above. As a result, the three additional parameters, $\Omega$, $\omega$ and $H$, which are required to extend coupled consolidation to unsaturated soil states, relate to four moduli, $E$, $E_w$, $R$ and $H$ rather than to three as in Wong et al. [33] ($E$, $R$ and $H$, as no distinction is made between $E_w$ and $H$);

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