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Lateral buckling of tapered members

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1. Introduction

The elastic lateral buckling of tapered I-section beams has been the subject of many investigations, but perhaps two of the more important are those of [1,2], which made theoretical and experimental investigations of doubly symmetric beams with tapered webs or flanges, and of mono-symmetric beams with tapered flanges (Fig. 1a). The flange bending during lateral buckling was analysed and used to establish the differential equilibrium equations for bending and torsion. Numerical solutions of these equations were obtained using the finite integral method [3], and compared with experimental results. While the finite integral method allowed faster convergence than the finite difference method which used to be in more common use, it is now rarely used.

Instead, tapered beams are now commonly analysed as a series of uniform elements, for which finite element computer programs [4,5] are widely available. An immediate problem with this approach is its inaccuracy, with significantly more elements required for tapered beam predictions of acceptable accuracy than for uniform beams. In addition, the uniform element method leads to conceptual difficulties with gross discontinuities occurring between elements. Further, it is possible that this method may converge on incorrect solutions. On the other hand, the use of tapered elements will hasten convergence, remove discontinuities between elements, and is expected to converge on accurate solutions.

There is therefore a need to develop a finite element method for tapered beams. Not only will such a method improve the ability to produce accurate solutions for the elastic buckling of tapered beams, it will also facilitate the analysis of the inelastic buckling of steel beams, in which the progressive yielding across a section and along a beam caused by the applied loads acting in conjunction with residual stresses cause the beam to become non-uniform and mono-symmetric. In the past, inelastic buckling has been analysed by reducing the elastic moduli in the yielded regions and performing an elastic buckling analysis [6]. More recently, it has been suggested [7–10] that a simple advanced method of determining the design resistance of a beam that is subject to lateral buckling may be developed by reducing the elastic moduli in accordance with basic design strength formulations and performing an elastic buckling analysis. While some analyses of this type have been made [11–13], they have used uniform elements, with the drawbacks discussed above. The development of a tapered beam finite element method can easily allow for linear variations in the elastic moduli along the element.

A further advantage of a tapered finite element is that it will facilitate explorations aimed at optimising the design of beams against lateral buckling [14] by varying the distribution of material along the beam length.

The purpose of this paper is to provide an alternate method of analysing the elastic lateral buckling of tapered beam-columns by using the energy method [15] to develop a finite element method [4] which can be applied to a wide range of loading and restraint conditions. Such a method starts from a consideration of the strain-deformation relationships and develops expressions for the strain energy stored and the work done during buckling. The following sections develop this alternate method.

It is conventional in member analysis to reduce the three-dimensional member to a one-dimensional line by replacing the member by a series of longitudinal fibres. The results of an analysis of a typical fibre are integrated across the section to derive the properties of a line which represents the aggregation of the fibres. For uniform members, it is natural to assume that these fibres are parallel to the centroidal and shear centre axes, but for tapered members this assumption leads to fibres which intersect the edges of the member's flange and web plates. This problem can be avoided by assuming [16,17] that the fibres form radial fans whose inclinations vary linearly between the web or flange boundaries. For mono-symmetric tapered members, this assumption leads to

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Nomenclature

A	area of cross-section	Q_Z, q_Z	concentrated and distributed loads parallel to Z axis
$A_{b,t,w}$	areas of bottom and top flanges and web	$t_{b,t,w}$	thicknesses of bottom and top flanges and web
$b_{b,t,w}$	widths of bottom and top flanges and web	$[T_{ieG,oeG}]$	matrices for transforming element to global deformations
$[B_{i,ou,0w}]$	matrices for generalised in-plane and out-of-plane strains	U,V,W	displacements of O in X, Y, Z directions
$[C_{i,ou,ow}]$	matrices for in-plane and out-of-plane nodal deformations	$U_{i,o}$	in-plane and out-of-plane strain energies
$[D_{i,ou,ow}]$	in-plane and out-of-plane stiffness matrices	U_P,V_P,W_P	displacements of P in X, Y, Z directions
$[D_{w,fuw,fv}]$	out-of-plane stability matrices	u_p, v_p, w_p	deflections perpendicular to and along inclined fibre
E,G	Young's and shear moduli of elasticity	W	work
$[G_o]$	global stability matrix	$w_{p,P}$	in-plane longitudinal deflections of inclined fibre
$[g_{oe}]$	element stability matrix	X,Y,Z	global coordinates measured from O
$I_{b,t}$	bottom and top flange second moments of area	x, y	principal axis coordinates
$I_{x,y}$	second moments of area about x, y axes	x_s, y_s	shear centre axis coordinates
I_{xw}	second moment of area of web	$Y_{c,s}$	Y coordinates of centroid and shear centre
J	uniform torsion section constant	$Y_{Q,q}$	Y coordinates of Q, q loads
$[K_{i,o}]$	global in-plane and out-of-plane stiffness matrices	Y_R, Y_r	Y coordinates of concentrated and distributed restraints
$[k_{ie,oe}]$	element in-plane and out-of-plane stiffness matrices	z, $z_{p,s}$	longitudinal axes from centroid, P, and shear centre
L	length of element or member	$\alpha_{f,t,w}$	ratios of minimum to maximum values of b_f, t_f, b_w
M	bending moment stress resultant	$\{\Delta_{i,o}\}$	global in-plane and out-of-plane nodal deformations
M_X	moment	$\{\delta_{ie,oe}\}$	element in-plane and out-of-plane nodal deformations
n	number of elements per half span	$\varepsilon_{i,o}$	in-plane and out-of-plane strains
N	axial tension stress resultant	$\{\varepsilon_{ou,ow}\}$	out-of-plane generalised strains
$Q_{ie,in}$	equivalent and actual nodal actions	Φ	rotation about Z axes
Q_Y, q_Y	concentrated and distributed loads parallel to Y axis	σ	normal stress
		λ	elastic buckling load factor

fibres that may intersect the distinct centroidal and shear centre axes, as shown in Fig. 1b.

2. Mono-symmetric tapered beam-columns

2.1. Section properties [18,19]

The mono-symmetric beam-column cross-section shown in Fig. 2 has bottom and top flange and web widths b_b, b_t, and b_w and thicknesses t_b, t_t, and t_w, respectively. The flange and web widths and thicknesses may be linearly tapered.

The areas and minor axis second moments of area are

$$A_b = b_b t_b, \quad A_t = b_t t_t, \quad A_w = b_w t_w,$$
$$A = A_b + A_t + A_w, \tag{1}$$
$$I_b = b_b^3 t_b/12, \quad I_t = b_t^3 t_t/12, \quad I_y = I_b + I_t$$

The beam-column has an arbitrary but convenient longitudinal axis OZ which is the locus of the web mid-heights as shown in Fig. 3, a perpendicular axis OY which coincides with the web mid-thickness line, and an axis OX perpendicular to the plane OYZ. (These axes allow the load and restraint positions to be defined without any prior calculations of the centroid and shear centre positions, and simplify both the continuity conditions between adjacent elements and the boundary conditions.)

The positions of the centroid C and shear centre S are defined by the Y distances from the OZ axis shown in Fig. 2

$$Y_c = \frac{b_w}{2}\frac{A_b - A_t}{A} \tag{2}$$

$$Y_s = \frac{b_w}{2}\frac{I_b - I_t}{I_y} \tag{3}$$

The major axis second moment of area is

$$I_x = \int_A (Y - Y_c)^2 dA = A_b y_b^2 + A_t y_t^2 + A_w Y_c^2 + b_w^3 t_w/12 \tag{4}$$

in which the centroidal coordinates y_b, y_t of the bottom and top flanges are given by

$$y_b = b_w/2 - Y_c$$
$$y_t = -b_w/2 - Y_c \tag{5}$$

The uniform torsion section constant is approximated by

$$J = (b_b t_b^3 + b_t t_t^3 + b_w t_w^3)/3 \tag{6}$$

2.2. Element model

The tangent to the locus of centroids defines the axis z shown in Fig. 3, and the perpendicular x, y axes are the principal axes of the cross-section. The tangent to the locus of shear centres defines the axis z_s shown in Fig. 3, and the x_s, y_s axes are perpendicular to the axis z_s. The centroidal z and shear centre z_s axes are inclined (at small angles Y_c' and Y_s') to the OZ axis, as indicated in Fig. 3.

The typical element of the tapered beam-column is modelled as being composed of tapered longitudinal fibres. The tangent to the fibre through a general point P(X,Y) in a cross-section defines the axis z_p as shown in Fig. 3. The fibre is inclined (at angles $X'$and Y') to the Z axis, as shown for a web fibre in Fig. 3. The fibres at the edges of the flanges and web coincide with these edges, and the inclinations of the interior fibres are assumed to vary linearly between those of the edges [16,17] according to

$$X'_{b,t} = X_{b,t} b'_{b,t}/b_{b,t} \tag{7}$$

and

$$Y' = Y b'_w/b_w \tag{8}$$

2.3. Loading and restraints

Concentrated transverse and longitudinal loads Q_Y (at Y_{QY}), Q_Z (at Y_{QZ}), and moments M_X may act at a section Z along the axis