Engineering Structures 150 (2017) 135–142

Contents lists available at ScienceDirect

Engineering Structures

journal homepage: www.elsevier.com/locate/engstruct

Enhancing the teaching of linear structural analysis using additive manufacturing

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ARTICLE INFO

Article history:
Received 7 March 2017
Revised 17 June 2017
Accepted 17 July 2017

Keywords:
Structural analysis
Flexural stiffness
Thermoplastic
3D printing
Education

ABSTRACT

Structural analysis forms a key component in many courses in civil, mechanical and aerospace engineering. Conventionally, the matrix stiffness method, a subset of finite element analysis, tends to occupy a central position in a typical syllabus, with a special focus on plane frames providing a bridge between basic structural components with pedagogical clarity and real-world structures. Equations of equilibrium are set-up and the full force of linear algebra brought to bear using the capabilities of Matlab or more specialized FEA packages. Such classes have a tendency to become a little dry and suffer from the usual shortcomings of numerical analysis and a black box approach - shortcomings in the sense of conceptual understanding as opposed to usefulness in the hands of experienced practitioners. The relatively recent advent of additive manufacturing is an exciting opportunity to incorporate a practical aspect to structural analysis. This paper describes the use of 3D printing, via the flexural stiffness of plane frames, to develop a structural feel for students, augmenting theoretical analyses. In addition to directly addressing the role of modeling, approximation, applicability of the underlying theory, and measurement uncertainty, it is thoroughly hands-on and initial anecdotal evidence suggests a higher degree of student buy-in.

1. Introduction

The bending stiffness of a structural member (based on standard beam theory) is a function of loading, boundary conditions, material and geometry [1]. For a given prismatic member the stiffness typically scales with EI/L^3, and it is easy to demonstrate the length dependency with a simple ruler for example. Analytically, given a set of forces and boundary conditions, we can integrate the governing expressions and obtain deflection, and hence stiffness. This process is not so straightforward for frames, consisting of an arrangement of beams (and columns) in typically rectangular combinations. However, the overall stiffness of such structures is very important and perhaps a key teaching opportunity occurs when we seek to shed light on how stiffness depends on these more realistic geometrical arrangements. This is where the stiffness method comes in.

Despite the liberating effect of Matlab [2] and the ease of numerical methods [3], the stiffness method becomes a decreasingly hands-on approach for all but the simplest examples. Many textbooks throughout the last 50 years or so have included chapters on the stiffness method including plane frames; a representative selection is [4–10]. However, it is possible to exploit versatile 3D printing for deepening an appreciation for structural behavior, and specifically in terms of printing a range of elastic, relatively slender, plane frames. Additionally, with relatively simple loading and boundary conditions it is then possible to assess the role of geometry and its influence on certain stiffness properties of the frames using measured data.

This paper considers the lateral stiffness of a baseline plane frame and some basic variations with three main foci:

- use of 3D printing in the teaching of structural analysis,
- exploring the influence of parameters on stiffness,
- assessing the role of simplifying assumptions.

The overall goal of the paper is to provide a systematic approach to developing a deeper student understanding of structural stiffness.

2. 3D printing of slender structures

3D printing has already revolutionized the teaching of *mechanical* engineering, via the rapid-prototyping of components like gears, for example. And given increasingly higher resolution it is now relatively easy to produce slender elastic elements and structures, and providing deflections and stresses are kept within

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http://dx.doi.org/10.1016/j.engstruct.2017.07.054
0141-0296/

acceptable limits, the behavior of such elements and structures is linear and elastic. Thus, 3D printing has potential as a teaching tool in the realm of deformable bodies and structures. Furthermore, in terms of the capabilities of 3D printing and elementary testing configurations, we shall focus attention on planar structural frames in which (i) the boundary conditions are either essentially fixed or free, and (ii) loads are applied at specific locations (point loads). This provides the context in which students can simply print and test structures and thus assess the important features of the flexural behavior in general, and aspects of the stiffness method in particular.

We shall focus attention on relatively flexible right-angled *plane* portal frames with clamped boundary conditions, moment-transmitting joints, and single point loading. This configuration was chosen partly to facilitate 3D printing and simple experimental testing, but it also represents a class of structure that is more instructive than a single structural element like a cantilever, and also can be analyzed using back-of-the-envelop (sway) calculations *in some cases*. Thus, in order to maintain a reasonable balance between hand-calculation and the fully fledged stiffness method we develop a focus on some simple portal frames, and compare theory (essentially the linear stiffness method using elastic beam elements) and experimental stiffness measurements.

More specific details will be given later, but by way of introduction, consider the middle frame shown in Fig. 1. If we clamp the lower edge, and subject the top corner to a lateral force, we can extract the stiffness of the frame in terms of lateral deflection. The stiffness of the columns scales with EI/L^3, and thus in moving to the frame shown on the left we expect a (much) greater stiffness, all other things being equal. Furthermore, if the cross-beam is relatively stiff (the frame on the right), we also expect an increase in stiffness, but in terms of reducing the rotation of the corners, and a deflected shape that resembles a lateral sway. This paper uses the 3D printer to produce frames that can be used to directly examines these effects.

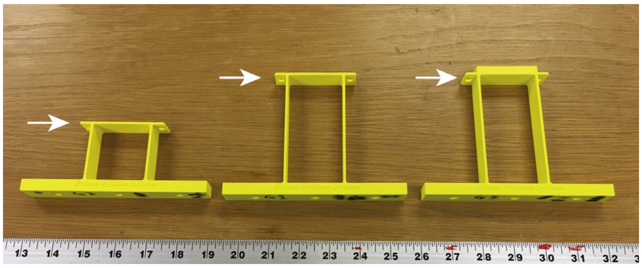

Fig. 1. Simple 3D-printed plane frames.

3. Flexural stiffness analysis of a simple portal frame

Historically, the teaching of structural analysis proceeded from pin-jointed trusses. This was partly justified by the ubiquity of riveted joints, but mostly because very often simplified analytical techniques, such as the methods of joints or sections, could be brought to bear [11]. But now most structural connections are rigid (e.g., welded) and computational techniques, for example the stiffness method, dominate. Thus frames rather than trusses now form the backbone of most courses on structural analysis (see the list of references). In order to explore this a little further consider a simple plane portal frame, consisting of two vertical columns that are clamped at their bottom ends and connected (via moment transmitting joints) to a horizontal beam as shown in Fig. 2(a). Assuming the lengths L and flexural rigidity EI are the same for all members, and that the frame is subject to a single horizontal load at one of the corners, we can write down the set of equilibrium equations for the structure:

$$\begin{bmatrix} F \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = EI/L^3 \begin{bmatrix} (AL^2/I+12) & 0 & 6L & -AL^2/I & 0 & 0 \\ & (AL^2/I+12) & 6L & 0 & -12 & 6L \\ & & 8L^2 & 0 & -6L & 2L^2 \\ & & & (AL^2/I+12) & 0 & 6L \\ & sym. & & & (AL^2/I+12) & -6L \\ & & & & & 8L^2 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ \theta_1 \\ X_2 \\ Y_2 \\ \theta_2 \end{bmatrix}, \tag{1}$$

in which (X, Y, θ) are global coordinates, relative to the bottom left corner of the frame. The element stiffness matrix for a beam-column in global coordinates can be found in Appendix A.

If we now assume an overall frame dimension (relevant to the physical dimensions to be 3D printed later) of $L = 0.1$ m, and a rectangular cross-section area $(b \times d) = (0.01 \times 0.002)$, gives an area $A = 20 \times 10^{-6}$ m^2, second moment of area $I = 6.67 \times 10^{-12}$ m^4, and thus $AL^2/I \equiv (L/r)^2 = 30 \times 10^3$, where r is the radius of gyration. These parameters relate to a geometry that may be considered highly flexible, and this facilitates relatively easy measurements. In analysis, the effective degrees of freedom can be reduced if we examine the relative magnitudes of each element and exploit certain symmetry conditions.

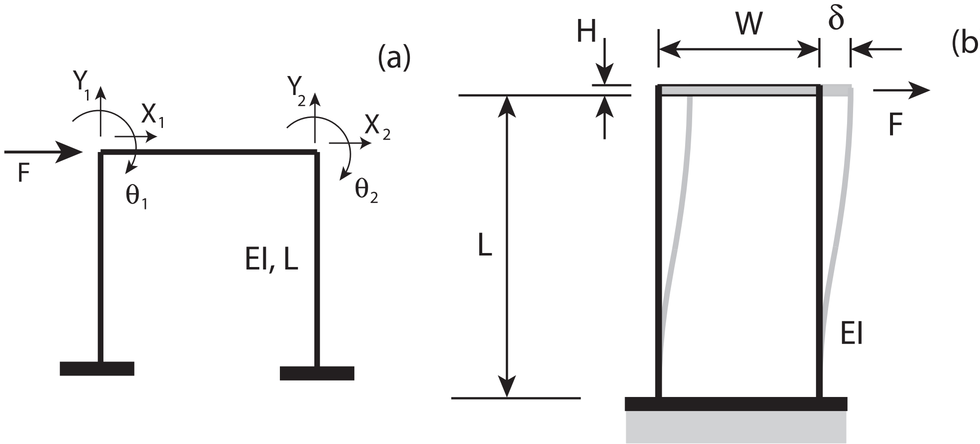

Fig. 2. (a) A simple square portal frame with identical beam and columns, (b) the baseline configuration.