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Semi-analytical solutions for optimal design of columns based on Hencky bar-chain model

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ABSTRACT

This paper is concerned with the shape optimization problem of columns for a given volume and length against buckling by using the discrete link-spring model or the so-called Hencky bar-chain model (HBM). This discrete beam model comprises a finite number of rigid segments connected by frictionless hinges and rotational springs. In particular, the rotational spring stiffness of HBM is a function of the square of cross-sectional area of columns with regular polygonal or circular cross-sectional shape. Therefore, the design of optimal rotational spring stiffnesses of a HBM allows one to obtain the optimal shape of a column provided that the assumed number of springs is sufficiently large. The present formulation of HBM for column optimization is prompted by some discrepancies in the volume calculations and the specification of the spring stiffness at the clamped end in Krishna and Ram (2007) discrete link-spring model formulation. By using the correct formulation and the semi-analytical method proposed by Krishna and Ram (2007), we determine the optimal shape of clamped-free, pinned-pinned, clamped-spring-supported columns. In addition, we extend the semi-analytical method to optimize the shape of clamped-free columns under distributed loads. Also presented herein are exact buckling solutions for the uniform HBM under axial load and selfweight as well as the non-uniform HBM under axial load with a specific class of spring stiffnesses.

1. Introduction

Proposed by Hencky [2], the so-called Hencky bar-chain model (HBM) discretizes the continuum Euler-Bernoulli beam into a finite number of rigid beam segments that are connected by frictionless hinges and rotational springs. The rotational springs are introduced to take care of the flexibility of beams. For a prismatic beam, the spring stiffness is $C = EI/a$, where EI is the flexural rigidity of the beam and a the segmental length. This formula can be readily obtained from the moment-curvature relationship. The advantage of HBM lies in the convenience for determining beam solutions by solving a set of algebraic equations instead of the differential governing equation.

In 1951, Salvadori [3] published a paper detailing how to use the central finite difference method for solving buckling problems of beams, plates and shells. In a discussion paper, Silverman [4] pointed out that the HBM is mathematically identical to the finite difference model (FDM) provided that the segmental length of the former model is made equal to the nodal spacing of the latter model. This equivalence was later proved by Leckie and Lindberg [5] for beam vibration problems. Based on this equivalence, Wang et al. [6] found that the end rotational spring stiffness C_R of HBM should take a form as $C_R = 2C/(1 + 2C/K_R)$ where K_R is the actual rotational spring stiffness for the beam elastic ends. Therefore, it is clear that C_R = (0, 2C) correspond to K_R = (0, ∞), where the bracketed values represent free rotation and no bending rotation, respectively. Hencky [2] also pointed out that the rotational spring stiffness of HBM should be 2C at the clamped end and this point was confirmed by El Naschie [7]. In fact, the HBM has an advantage over the FDM in that the HBM does not have any fictitious joints and each joint of HBM has a clear physical meaning.

Recently, HBM was further developed to handle the buckling problems of non-uniform columns [8], prismatic columns under selfweight [9] and the buckling and vibration problems of non-

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uniform columns resting on partial Winker foundation [10]. In addition to solving continuum beam problems, one promising application of HBM is for analysis of articulated beams or beam structures with repetitive cells, such as railroad tracks, pavements, pontoon type bridges or spacing structures comprising many modules [11]. This is because HBM is a natural model for such aforementioned structures. A HBM planar grid may also be used to model 2D planar structures [12–15]. Another interesting application of HBM is that it could be used to calibrate Eringen's small length scale coefficient due to the phenomenological similarities between the HBM and the Eringen's nonlocal beam model. This application was studied in papers [9,16–24].

Naturally, one questions whether the HBM can be used to optimize the shape of columns? The history of optimal design of columns for a given volume and length against buckling begins with Lagrange's unsuccessful attempt in 1773 [25,26]. The correct optimization for a simply-supported column with circular cross-section was formulated by Clausen [27]. Keller [28] later obtained the analytical optimal solution for the simply-supported column with arbitrary convex cross-section. The analytical solutions for columns with clamped-clamped, clamped-free and clamped-pinned ends were further obtained by Tadjbakhsh and Keller [26]. However, Olhoff and Rasmussen [29] found that the optimal buckling loads may be bimodal or unimodal, which is dependent on the value of the prescribed minimal cross-sectional area of columns with clamped ends. More recent papers on column optimization were written by Gil-Martín et al. [30], Maalawi [31], Novakovic and Atanackovic [32], Polajnar et al. [33], Bochenek and Tajs-Zielińska [34], Wang et al. [35].

The shape optimization for columns using HBM (or discrete link-spring model) was first studied by Prager and Prager [36]. In their paper, they only used five rigid segments to model the column and considered the constraint that the sum of internal rotational spring stiffnesses is a constant. More recently, Krishna and Ram [1] optimized the shape of columns using HBM with more segments and considered the volume constraint that requires the sum of the square root of the stiffnesses be kept as a constant. Krishna and Ram [1] proposed a semi-analytical method which allows the cross-sectional areas at the spring locations to be determined analytically without any prescribed function for the area variation along the beam length. This method also overcomes a shortcoming of the finite element method that requires resizing of the elements and recomputing their stiffness properties during the optimization process [37]. However, Krishna and Ram [1] did not set the end rotational spring stiffness $C_R = 2C$ at the clamped end, but took C_R as infinitely large. Furthermore, the cross-sectional area for the free or pinned end was assumed to be zero since the rotational spring stiffness at such boundary condition is zero, which may be not practical. Also, there is a question on the assumption of using a small number of rotational springs to interpret the varying column shape, which may produce an inaccurate higher bucking load of HBM than that of the continuum column because of the presence of rigid segments. This study will discuss these refinements in Krishna and Ram's formulation and solutions as well as extend the method for solving the optimal shape of columns under distributed load.

In the next section, the general semi-analytical formulation for columns under constant axial load is presented. In Section 3, clamped-free, pinned-pinned and clamped-spring-supported columns treated by Krishna and Ram [1] are re-analyzed. Section 4 deals with the optimal shape of clamped-free column under distributed load obtained by using the semi-analytical method. The analytical buckling loads for uniform HBM under axial load and selfweight are derived in Section 5 and those for non-uniform HBM under axial load with a specific class of spring stiffnesses are presented in Section 6.

2. Semi-analytical formulation of optimization for column under axial load

Consider a HBM of n rigid segments with equal segmental length $a = L/n$, subjected to a constant axial compressive load P as shown in Fig. 1a. The rotational angle at the elastic joint j is denoted by θ_j with $j = 0, \ldots, n-1$. The segments are connected by frictionless hinges and rotational springs having stiffness $C_j = EI_j/a$ for joint $j = 1, \ldots, n-1$, where E is the Young's modulus and I_j the second moment of area at joint j. The internal spring stiffnesses of the HBMs are not the same since we are modelling a non-uniform column. The bottom end A (the origin of HBM, i.e. $x = 0$) is rigidly supported and the rotational springs having stiffnesses C_{RA}, C_{RB} and a lateral spring C_{LB} are presented to accommodate different boundary restraints for the column.

For simplicity, we shall assume a circular cross-section and let A_j denote the area of the jth segment of the column. Accordingly, the internal rotational spring stiffness C_j of the HBM is related to A_j by

$$C_j = \frac{EI_j}{a} = \frac{EA_j^2}{4\pi a} \quad \text{for } j = 0, \ldots, n \tag{1}$$

Fig. 1b shows the piecewise constant column shape that is assumed to correspond to the HBM in Fig. 1a. It must be emphasized that the piecewise constant column (for approximating the continuously varying cross-section column) shown in Fig. 1b is an interpretation of the HBM and it is only accurate when we take sufficient number of segments, say $n = 100$. In the figure, $r_0, r_1, \ldots, r_{n-1}, r_n$ are the radii of the piecewise constant segments and the radii are associated with the rotational spring stiffnesses $C_0, C_1, \ldots, C_{n-1}, C_n$ respectively. As shown in Fig. 1b, the length of the two end segments are assumed to be $a/2$ whereas the length of the internal segments is a. Consequently, the volume of the assumed piecewise constant column is given by

$$V = a\sum_{j=1}^{n-1} A_j + \frac{1}{2}aA_0 + \frac{1}{2}aA_n \tag{2}$$

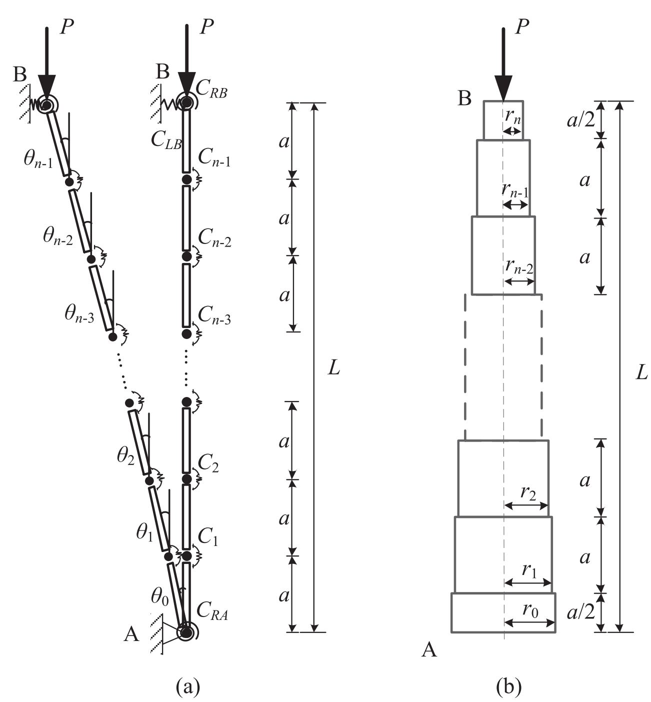

Fig. 1. (a) General HBM under axial compressive load P and (b) corresponding column shape.