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Critical examination of midplane and neutral plane formulations for vibration analysis of FGM beams

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ABSTRACT

There has been a controversial claim that the beam model formulation for functionally graded materials (FGM) beams must be based on the neutral plane for correct solutions. This claim cuts across well accepted mid-plane formulation for FGM beams. Presented herein is a critical examination of the mid-plane and neutral plane formulations for the vibration analysis of FGM beams. It will be shown herein that the problem arises from a misconception that the immovable supporting points are located at the mid-plane when the supporting points are actually at the neutral plane when basing the formulation on the neutral plane. The positioning of the immovable simple supports at two different planes leads to the difference in results. However, in the case of movable simple supports, the mid-plane formulation furnishes the same vibration solutions as the neutral plane formulation, even though the supports are located on different planes. Both formulations furnish the same frequency results for clamped ends. The conclusion is that there is nothing wrong with using the mid-plane formulation for FGM beams. In fact, by using the neutral plane formulation, it would be difficult to solve FGM beams with a constraint on the longitudinal displacement at the midplane.

1. Introduction

Midplane formulation is most commonly used to analyze functionally graded beams or plates [1–9]. However, several researchers have suggested that the neutral plane formulation must be used, instead of the mid-plane formulation, for analysis of functionally graded materials (FGM) beams. Abrate [10] demonstrated that elastic coupling due to variation in material properties through the thickness of plate can be eliminated by selecting the neutral plane as the reference plane for the formulation of the plate equations. By doing so, the formulation of the FGM plate becomes similar to that of a homogeneous plate. They also claimed that the neutral plane formulation is applicable for nonlinear geometric problems. Zhang and Zhou [11] concluded that the use of the neutral plane for the thin plate theory has more merits in engineering applications because it is simpler than classical laminated plate theory that is based on middle surface. Yaghoobi and Fereidoon [12] performed bending analysis of simply supported FGM beams under uniformly distributed load by using the neutral plane formulation. They showed that the deflections obtained are larger than their counterparts based on the midplane formulation. Larbi et al. [13] developed a new shear deformation beam theory based on the neutral axis being the reference axis and calculated the vibration frequencies of simply supported beam where the ends were movable in the axial direction. They showed that the vibration frequencies calculated from their shear deformation theory are in very close agreement with the vibration frequencies obtained from other shear deformation theories that rely on the mid-plane formulation. Eltaher et al. [14] adopted the neutral plane for the Euler Bernoulli beam formulation to calculate the vibration frequencies of FGM beams and compared the results with those obtained using mid-plane formulation. Their calculations showed that the vibration frequencies obtained from mid-plane and neutral plane formulations are different; by as much as 10% for certain values of gradient index. Yin et al. [15] claimed that the midplane formulation is not suitable for vibration analysis of FGM plates and that the neutral plane formulation must be employed instead. Zhang [16] modified Reddy's third order beam theory (Reddy [17]) to account for the effect of neutral plane and employed the modified third order beam theory to study post-buckling and nonlinear vibration responses of FGM beams. They

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also showed that the neutral plane formulation gives rise to simpler governing equations for FGM beams. Eltaher et al. [18] studied the bending and buckling problems of modified functionally graded nanobeams modeled by Timoshenko beam theory and they showed that the deflections and buckling loads calculated from the midplane and neutral plane formulations may differ by as much as 16% for certain cases and concluded that neutral formulation must be used for FGM beam analysis. On the other hand, Zoubida et al. [19] demonstrated that the mid-plane based and the neutral plane based formulations furnish identical linear vibration frequencies. Lee et al. [20] studied the thermal buckling behavior of functionally graded plates and claimed that the neutral plane formulation predicts a smaller critical temperature when compared to the mid-plane formulation. It can be seen that the conclusion arrived at by various aforementioned researchers on the use of neutral plane formulation is rather discomforting as many researchers have adopted the midplane formulation for their analysis of FGM beams and plates.

Prompted by the existing confusion, this study critically examines the mid-plane and neutral plane formulations in the context of linear vibration of FGM beams. If the claim made by some researchers that the neutral plane is the correct formulation, then it would mean that all previous works done on functionally graded beams, antisymmetric laminated beams and any beam with non-symmetric properties based on the mid-plane formulation have predicted erroneous results. On the other hand, if the claim is not true, where are the misconceptions?

2. Formulation based on mid-plane for linear vibration of FGM beams

First, we present the mid-plane formulation of the Euler-Bernoulli beam model for FGM beams. Consider a functionally graded beam with length L, thickness h and Cartesian x–z coordinate system where the origin is at the left end of the beam as shown in Fig. 1. The FGM is a mixture of ceramics and metal. Its Young's modulus $E(z)$, mass density $\rho(z)$ and Poisson's ratio $\nu(z)$ are assumed to change continuously along the thickness according to the power law distribution

$$E(z) = (E_2 - E_1)\left(\frac{2z+h}{2h}\right)^n + E_1 \tag{1}$$

$$\rho(z) = (\rho_2 - \rho_1)\left(\frac{2z+h}{2h}\right)^n + \rho_1 \tag{2}$$

$$\nu(z) = (\nu_2 - \nu_1)\left(\frac{2z+h}{2h}\right)^n + \nu_1 \tag{3}$$

where the subscripts "1" and "2" denote the metal and ceramic constituents, respectively. The top surface $(z=-h/2)$ of the beam is 100% metal while the bottom surface $(z=h/2)$ is 100% ceramics. The power n is the gradient index characterizing the distributions of the material properties. When $n=0$, the FGM beam reduces to a homogeneous beam.

According to the Euler-Bernoulli beam theory, the longitudinal displacement $u(x,z,t)$ and the transverse displacement $w(x,z,t)$ are, respectively, given by

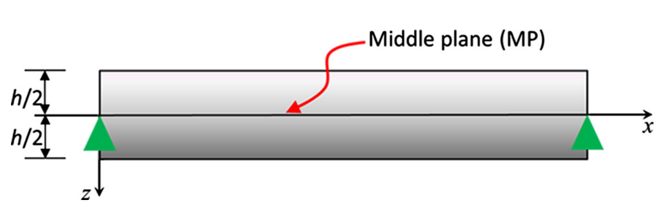

Fig. 1. FGM beam with mid-plane formulation.

$$u(x,z,t) = u_0(x,t) - z\frac{\partial w_0}{\partial x}, \tag{4}$$

$$w(x,z,t) = w_0(x,t) \tag{5}$$

where $u_o(x,t)$ and $w_o(x,t)$ are the displacement components on the mid-plane. In view of the longitudinal displacement, the normal strain ε_x is given by

$$\varepsilon_x = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} \tag{6}$$

By assuming an elastic constitutive relation, the normal stress σ_x is given by

$$\sigma_x = \frac{E(z)}{1-\nu^2(z)}\left[\frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2}\right] \tag{7}$$

Hamilton's principle requires that

$$\int_0^T (\delta U - \delta T)dt = 0 \tag{8}$$

For convenience, the width of the beam is assumed to be unity so that the integrals over the cross-section area may be written as an integral over the depth of the beam. Thus, the variational strain energy δU is given by

$$\begin{aligned}\delta U &= \int_{x=0}^{L}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x\left[\frac{\partial \delta u_0}{\partial x} - z\frac{\partial^2 \delta w_0}{\partial x^2}\right]dzdx = \int_{x=0}^{L}\left[N_x\frac{\partial \delta u_0}{\partial x} - M_x\frac{\partial^2 \delta w_0}{\partial x^2}\right]dx \\ &= \int_{x=0}^{L} -\left[\frac{\partial N_x}{\partial x}\delta u_0 + \frac{\partial^2 M_x}{\partial x^2}\delta w_0\right]dx + [N_x\delta u_0]\big|_{x=0,L} - \left[M_x\frac{\partial \delta w_0}{\partial x}\right]\Big|_{x=0,L} \\ &\quad + \left[\frac{\partial M_x}{\partial x}\delta w_0\right]\Big|_{x=0,L}\end{aligned} \tag{9}$$

and the variational kinetic energy δT is given by

$$\begin{aligned}\delta T &= \int_{x=0}^{L}\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)\left(\dot{u}_0 - z\frac{\partial \dot{w}_0}{\partial x}\right)\left(\delta\dot{u}_0 - z\frac{\partial \delta\dot{w}_0}{\partial x}\right)dzdx \\ &\quad + \int_{x=0}^{L}\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)\dot{w}_0\delta\dot{w}_0 dzdx \\ &= -\int_0^L I_0\ddot{u}_0\delta u_0 dx + \int_0^L I_2\frac{\partial^2 \ddot{w}_0}{\partial x^2}\delta w_0 dx - \int_0^L I_1\frac{\partial \ddot{u}_0}{\partial x}\delta w_0 dx \\ &\quad + \int_0^L I_1\frac{\partial \ddot{w}_0}{\partial x}\delta u_0 dx - \int_0^L I_0\ddot{w}_0\delta w_0 dx\end{aligned} \tag{10}$$

Note that Eq. (10) is simplified by setting the integrals to zero that are evaluated at $t=0$ and $t=T$ because the virtual displacements δu_0 and δw_0 are zero at $t=0$ and $t=T$.

By gathering coefficients of the variational terms δu_0, δw_0 and by setting these coefficients to zero, we derive the following equations of motion

$$\frac{\partial N_x}{\partial x} = I_0\frac{\partial^2 u_0}{\partial t^2} - I_1\frac{\partial^3 w_0}{\partial x\partial t^2} \tag{11}$$

$$\frac{\partial^2 M_x}{\partial x^2} = I_0\frac{\partial^2 w_0}{\partial t^2} + I_1\frac{\partial^3 u_0}{\partial x\partial t^2} - I_2\frac{\partial^4 w_0}{\partial x^2\partial t^2} \tag{12}$$

In view of Eq. (7), the force-displacement relations are given by

$$N_x = A_{11}\frac{\partial u_0}{\partial x} - B_{11}\frac{\partial^2 w_0}{\partial x^2} \tag{13}$$

$$M_x = B_{11}\frac{\partial u_0}{\partial x} - D_{11}\frac{\partial^2 w_0}{\partial x^2} \tag{14}$$

where $I_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}}\rho(z)dz$, $I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\rho(z)dz$, $I_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2\rho(z)dz$, $A_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{E(z)}{1-\nu^2(z)}dz$, $B_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{zE(z)}{1-\nu^2(z)}dz$, $D_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{z^2E(z)}{1-\nu^2(z)}dz$.

By substituting Eqs. (13) and (14) into Eqs. (11) and (12), the equations of motion may be rewritten as

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} - B_{11}\frac{\partial^3 w_0}{\partial x^3} = I_0\frac{\partial^2 u_0}{\partial t^2} - I_1\frac{\partial^3 w_0}{\partial x\partial t^2} \tag{15}$$

$$B_{11}\frac{\partial^3 u_0}{\partial x^3} - D_{11}\frac{\partial^4 w_0}{\partial x^4} = I_0\frac{\partial^2 w_0}{\partial t^2} + I_1\frac{\partial^3 u_0}{\partial x\partial t^2} - I_2\frac{\partial^4 w_0}{\partial x^2\partial t^2} \tag{16}$$