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Closed-form moment-curvature relations for reinforced concrete cross sections under bending moment and axial force

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ABSTRACT

The paper develops a computer strategy for the analysis of cracked reinforced concrete cross sections within the scope of Eurocode 2, using a symbolic algebraic manipulator. The nonlinear constitutive behaviours for either concrete or steel are considered. For concrete, a nonlinear constitutive equation, function of the concrete class, is adopted, as indicated in Eurocode 2 for structural analysis. This equation is more general than the parabola equation used in many models for both ultimate and service design. For the steel, the present model uses a continuous and smooth function that approximates both the elastic and plastic ranges with hardening, instead of the traditional multi-linear forms. The moment-curvature relations are obtained by imposing equilibria for the bending moments and longitudinal forces, assuming that cross sections remain plane after deformation and that there is a perfect adherence between concrete and steel. In addition, closed-form solutions are also proposed and the cross section's collapse and rotation capacity is determined by the reaching of the ultimate strains for concrete or steel. The paper ends with the presentation of illustrative examples, for rectangular and T- cross sections.

1. Introduction

The structural design of long prismatic reinforced concrete beams and columns requires a deep analysis of the cross-section behaviour by means of moment M-curvature ϕ relations [1–3]. These relations are very important for several relevant phenomena, such as strength and rotation capacity, hysteretic behaviour and stability effects, and are affected by the highly nonlinear constitutive laws for both concrete and steel [4]. In fact, the concrete's constitutive relation is highly nonlinear and is not even symmetrical, since concrete's strength in tension is much smaller than in compression – when the tensile stresses in concrete surpass its tensile strength, the cross section must be considered as fractured and the tensile stresses are supported only by the steel rebars. On the other hand, the steel behaviour is assumed to be symmetrical both in tension and compression, but the relation becomes nonlinear after reaching the yielding stress. Usually, the modelling of such behaviour is made by means of multilinear functions, which are cumbersome to handle in computer modelling. The objective of the present paper is thus to study the moment-curvature of concrete cross sections with various shapes, submitted to both bending moments and axial forces and taking into account the materials nonlinearities in their full form, as described by the Eurocode 2 [5]. We will not concern ourselves with shear deformations, since we are dealing with long prismatic members and the reinforced concrete's behaviour is analysed at the cross section's level. We will place our focus on the development of numerical strategies for nonlinear analysis, using the large capacities of the contemporary symbolic programing packages such as Mathematica [6].

For several decades, this theme has been object of much research by many groups worldwide, due to its relevance in the construction's practice. As examples, we refer the work of Cohn and Ghosh [7], devoted to the development of moment-curvature relations taking into account the effects of the steel's plastic strains on the ductility behaviour or reinforced concrete structures, and also the paper of Bathe et al. [8], where the nonlinear analysis of reinforced concrete structures is devoted to finite element three dimensional analysis and a tension-stiffening model is also adopted. The nonlinear behaviour of reinforced concrete sections under combined bending moments and axial forces is also addressed in Riva and Cohn [9], where the ultimate design was analysed considering a simplified stress-strain law for the concrete in compression. Shortly afterwards, these same authors [10]

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applied a lumped-plasticity strategy for the nonlinear structural analysis of reinforced concrete structures, developing moment-curvature constitutive relations that used realistic material laws. In the context of the Finite Elements Method, it is worth to refer also the work of Spacone et al. [11] that considers the inelastic behaviour of reinforced concrete and models the flexural behaviour of the element in a differential form of the moment curvature relation by means of a Maxwell model. He et al. [12] extended the analysis to reinforced concrete sections with polymer rebars and, which show low modulus of elasticity, and a trilinear moment curvature relationship was developed to evaluate the flexural cross-section response. Torrico [13] considered a linearization of the moment-curvature relation to obtain the post-critical behaviour of high strength reinforced concrete columns, including the geometrical and material nonlinearities and the effects of confinement due to the reinforcement. Also on the buckling response of reinforced concrete members, Challamel and Hellesland [14] adopted a moment-curvature law based on continuum damage mechanics theory, and Picandet et al. [15] adopted a bi-linear moment-curvature relation to model the instability of compressed columns with both geometric and material non-linearities for several boundary conditions. Ali et al. [16] related to the discrete rotation in the cracked sections. Recently, the ultimate design of multi-rectangle reinforced concrete sections in closed form was developed in Silva et al. [17], while Chandrasekaran et al. [18] were focused on the ductility capacity of reinforced concrete sections and developed closed-form formulae for use in the engineering practice.

In the present work, the analytical moment-curvature (M-ϕ) curves are derived, and it was considered that concrete cracks under tension and the nonlinear constitutive laws for the compressed concrete, defined precisely by the expression proposed in the Eurocode 2 [5], are applied in their full form. In addition, the steel behaviour is modelled by a single and continuous function that approximates as closely as wished the bi-linear law presented in Eurocode 2 [5]. So, instead of the commonly adopted multilinear formulations, the nonlinear constitutive behaviour of steel was modelled here by a single expression, by means of the Goldberg-Richard formula [19,20]. This strategy is able to account very easily for steel hardening effects and betters significantly the computational efficiency of the analysis. A computer program is developed, in the context of the modern symbolic package Mathematica [6], and a parametric study is presented. It highlights the influence of several relevant aspects on the moment-curvature relations, such as the axial force and the compression steel percentage, and is illustrated by plotting the bending moment or the depth of the neutral axis against the section's curvature. Finally, since all deformations are monitored, the analysis is able to predict cross section collapse due to the reach of the maximum admissible strains either in the steel rebars or in the concrete part. The enhancements are illustrated by the analyses of two cross sections with distinct shapes (a rectangular section and a T-section) for the cases of pure bending and combined bending moment and axial force. Since the model monitors the strains in concrete and steel rebars, it is able to predict the rotation capacity and the section failure. The moment curvature curves obtained are also compared to closed-form approximate expressions. A set of conclusions ends the paper.

2. The nonlinear reinforced concrete cross section analysis

2.1. General assumptions

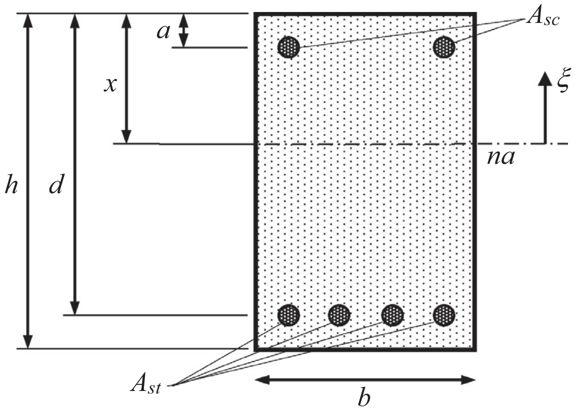

Fig. 1. Generic rectangular RC cross section.

A generic rectangular reinforced concrete (RC) section is illustrated in Fig. 1. The cross section has the following geometric properties: width b, height h, concrete cover a (measured from the steel centre to the concrete face) and effective height $d = h - a$. The areas of steel in tension and in compression are denoted by A_{st} and A_{sc}, respectively. Variable x represents the depth of the neutral axis (na), ξ is the transverse coordinate and we suppose, without loss of generality for the purposes of the present work, that the cross section is submitted only to axial forces and positive bending moments along the neutral axis, generalization to negative bending moments being trivial.

2.1.1. The stress-strain relations for concrete

For a generic concrete fibre submitted to the shortening strain ε_c at the generic point with transverse coordinate ξ, the correspondent compressive stress is given by expression (see [5]):

$$\sigma_c(\xi) = f_{cm} \cdot \frac{k \cdot \frac{\varepsilon_c(\xi)}{\varepsilon_{c1}} - \left(\frac{\varepsilon_c(\xi)}{\varepsilon_{c1}}\right)^2}{1 + (k-2)\frac{\varepsilon_c(\xi)}{\varepsilon_{c1}}} \tag{1}$$

where parameter k is given by

$$k = 1.05 \cdot \frac{E_{cm} \cdot \varepsilon_{c1}}{f_{cm}} \tag{2}$$

and ε_{c1} is the shortening concrete strain at the peak stress. f_{cm} and E_{cm} are respectively the mean values of concrete compressive strength and modulus of elasticity. We neglect the concrete tensile strength and Fig. 2 illustrates the compressive stress law (1) for several concrete classes. It highlights the decrease of ductility for higher strength concrete classes, whose ultimate strain is $\varepsilon_{cu} = 2.8‰$ for C70 and C90 instead of $\varepsilon_{cu} = 3.5‰$, the usual value for normal strength concrete classes C12 to C50. In addition, the concrete's behaviour becomes more nonlinear for the lower concrete classes. Fig. 3 shows the same constitutive laws for the same

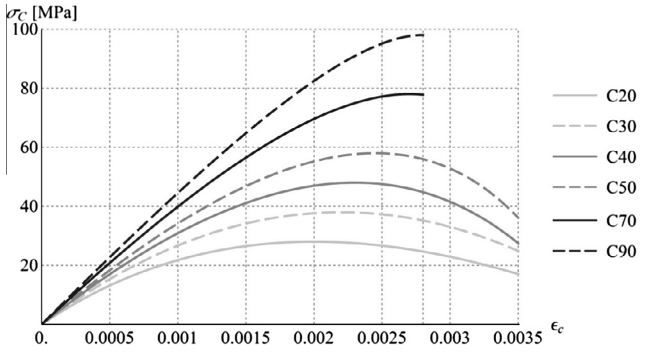

Fig. 2. Constitutive laws for various concrete classes.

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