Engineering Structures 126 (2016) 252–263

Contents lists available at ScienceDirect

Engineering Structures

journal homepage: www.elsevier.com/locate/engstruct

Hencky bar-chain model for buckling and vibration analyses of non-uniform beams on variable elastic foundation

H. Zhang [a,*], C.M. Wang [b], E. Ruocco [c], N. Challamel [d]

[a] Department of Civil and Environmental Engineering, National University of Singapore, Kent Ridge, Singapore 119260, Singapore
[b] Engineering Science Programme and Department of Civil and Environmental Engineering, National University of Singapore, Kent Ridge, Singapore 119260, Singapore
[c] Department of Ingegneria Civile, Design, Edilizia e Ambiente, Second University of Naples, Via Roma 29, 81031 Aversa, CE, Italy
[d] Université de Bretagne Sud, UBS – Institut de Recherche Dupuy de Lôme (IRDL), Centre de Recherche, Rue de Saint Maudé, BP92116, 56321 Lorient cedex, France

ARTICLE INFO

Article history:
Received 5 March 2016
Revised 28 July 2016
Accepted 31 July 2016

Keywords:
Buckling
Vibration
Hencky bar-chain
Central finite difference
Non-uniform beam
Partial variable elastic foundation

ABSTRACT

This paper is concerned with the development of the Hencky bar-chain model (HBM) for the buckling and vibration analyses of non-uniform beams resting on partial variable elastic foundation. The HBM allows analysts to obtain beam buckling and vibration solutions by solving a set of algebraic equations instead of a differential equation. To get the HBM, we resort to the first order central finite difference model (FDM) because both discrete models have phenomenological similarities. Based on bending moment, shear force and deflection equivalence between the two discrete beam models, the expressions for the internal spring stiffness, the end spring stiffness and the elastic foundation stiffness for the HBM are obtained. Some example problems are considered to demonstrate the simplicity of the HBM for buckling and vibration analyses of non-uniform beams on partial variable elastic foundation by taking the number of segments of the HBM to infinity. The effects of the foundation length and stiffness on the buckling load and natural frequencies are also discussed. Naturally, the HBM can be used to obtain solutions for articulated beams or beam-like structures with repetitive cells as well.

1. Introduction

Non-uniform columns are commonly used in structural buildings, machine and aeronautical space structures to lighten the weight, save material and even for aesthetic reasons. The buckling and vibration problems of beams and columns resting on Winkler elastic foundation have been studied before. But most studies are restricted to either non-uniform beams on full elastic foundation or uniform beams on partial elastic foundation. For example, Hsu [1] investigated the vibration problem of non-uniform beam with full elastic support using the spline collocation method. Girgin and Girgin [2] derived the static or dynamic stiffness matrix of non-uniform beam members resting on a full variable elastic foundation using the Mohr method. The dynamic analysis of non-uniform beam resting on full Winkler elastic foundation was conducted by Lee and Ke [3] and Abohadima and Taha [4]. But the effect of the partial elastic foundation was not taken into account in these papers. Pavlović and Tsikkos [5], Doyle and Pavlovic [6], Eisenberger et al. [7,8] and Eisenberger and Clastornik [9] studied the bending, buckling and vibration problems of Euler-Bernoulli beams on partial elastic foundation and Yokoyama [10] performed a similar analysis for Timoshenko beams. However, the abovementioned research papers focussed on uniform beams on partial elastic foundation. This paper treats the more general beam buckling and vibration problems that allow for partial variable elastic foundation, varying beam cross-section and density as well as general elastic end restraints.

In the past years, there were many papers written on the buckling and vibration analyses of non-uniform columns. Different methods for analyses include the finite element method [11–14], the Rayleigh–Ritz method [15,16], the differential quadrature method [17], the Bessel function approach [18,19], the power function approach [19], the integral-equation approach [20], the imperfection method [21], and the Haar wavelet approach [22]. However, the modelling of HBM for the non-uniform beams on variable elastic foundation in the context of buckling and vibration problems has never been studied before.

The so-called Hencky bar-chain model was originally proposed by Hencky [23] to replace an Euler continuum beam model so that one can obtain solutions by solving a set of algebraic equations instead of a differential equation. It comprises rigid beam segments

* Corresponding author.
E-mail addresses: zhanghong@u.nus.edu (H. Zhang), ceewcm@nus.edu.sg (C.M. Wang), Eugenio.RUOCCO@unina2.it (E. Ruocco), noel.challamel@univ-ubs.fr (N. Challamel).

http://dx.doi.org/10.1016/j.engstruct.2016.07.062
0141-0296/

of equal length a (=L/n where L is the length of beam and n the segmental number) and connected to one another by frictionless hinges and rotational springs with stiffness $C = EI/a$ where EI is the flexural rigidity of the Euler beam. The total mass of the beam is distributed as lumped masses at the junctions with ρa for internal joints and $\rho a/2$ for the ends where ρ is the mass per unit length of the beam.

It was Silverman [24] who first pointed out the equivalence between the HBM and the FDM for beam buckling problem as long as the segmental length of HBM is equal to the nodal spacing of FDM. The same analogy was later mentioned by Leckie and Lindberg [25] for beam vibration problems. With the same value of segmental length and nodal spacing, the governing equations of the two models are mathematically the same for the internal beam domain. However, in order to keep the HBM exactly analogous to the FDM, the rotational spring stiffnesses at the beam ends have to be modelled appropriately. Hencky [23] pointed out that the rotational spring stiffness has to take on $2C$ for a clamped end which was also proven by El Naschie [26] and Ruocco et al. [27]. More recently, Wang et al. [28] showed that the Hencky's rotational spring stiffness takes on the form of $2C/(1 + 2C/K)$ for an elastic end with rotational stiffness of K. According to the above research work, the HBM may be regarded as a physical structural model for the finite difference equations for beams.

The HBM has been fully developed for uniform beams with general end restraints. In order to simulate non-uniform beams on variable elastic foundation, the original HBM has to be modified. The rotational spring stiffnesses and lumped masses need to be reformulated. Moreover, the new boundary conditions for HBM due to the partial elastic foundation need to be derived. It is therefore the aim of this paper to present a more general HBM for application to buckling and vibration analyses of non-uniform beams with end springs and partial variable elastic foundation.

Originally meant to replace the continuum beam, the HBM can be regarded as the exact model for articulated periodic beam structures by adopting the appropriate spring siffnesses at the joints. Such articulated periodic beam structures find applications as pipelines, trains consisting of many cars, multi-span floating bridges or space structures comprising many modules [29]. In physics, the HBM acts as the small-scale periodic chain of mass and spring in lattice dynamics. Therefore, the HBM may be used to calibrate the Eringen's small length scale coefficient as suggested by Challamel et al. [30–33], Duan et al. [34] and Wang et al. [35,36] based on the phenomenological similarities of the HBM and the Eringen's nonlocal beam model.

In this paper, the next two sections will present detailed derivations of FDM and HBM for non-uniform beams on partial variable elastic foundation in the context of buckling and vibration problems. Next, by resorting to the equivalence of HBM and FDM, the expressions for the spring stiffnesses of HBM are obtained. Some example problems will be presented to show the convenience and accuracy of the HBM. The effect of partial variable elastic foundation on the buckling load and natural frequencies of non-uniform beams will also be discussed.

2. FDM for non-uniform beam

In order to develop the HBM, we resort to the FDM. Consider an Euler-Bernoulli beam of length L, flexural rigidity $EI(x)$, mass per unit length $\rho(x)$ subjected to an axial compressive load P. The beam is resting on a partial variable elastic foundation of length qL and stiffness intensity $k(x)$. The left end A and right end B are restrained by rotational springs having stiffnesses K_{RA}, K_{RB} and lateral springs having stiffnesses K_{LA}, K_{LB} as shown in Fig. 1a. Fig. 1b shows the n-segmented FDM for the non-uniform beam with equal nodal spacing $a = L/n$ and node number from 0 to n. Therefore, the longitudinal coordinate of the j-th node is $x_j = ja$. In order to solve the fourth-order differential governing equation of the beam, four boundary conditions for each boundary point have to be introduced. For handling these boundary conditions, the central finite difference method is applied on the boundary point A, B and D. Particularly, four fictitious nodes -2, -1, $i+1$, $i+2$ for the first part while $i-2$, $i-1$, $n+1$, $n+2$ for the second part are created where $i = qn$.

The governing equation for the vibration problem of non-uniform beam under a compressive axial load P and resting on a partial variable elastic foundation is given by [19]

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w}{dx^2}\right) + P\frac{d^2w}{dx^2} - \rho(x)\omega^2 w + k(x)w = 0 \quad \text{for } 0 < x < qL \tag{1a}$$

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2w}{dx^2}\right) + P\frac{d^2w}{dx^2} - \rho(x)\omega^2 w = 0 \quad \text{for } qL < x < L \tag{1b}$$

where w is the transverse displacement, x the longitudinal coordinate with its origin at end A of the beam and ω is the angular vibration frequency of the beam. The boundary conditions for the elastically restrained ends are given by [19]

$$\left[\frac{d}{dx}\left(EI(x)\frac{d^2w}{dx^2}\right)\right]_{x=0} + P\left[\frac{dw}{dx}\right]_{x=0} + K_{LA}w(0) = 0 \quad \text{at } x = 0 \tag{2a}$$

$$EI(x)\left[\frac{d^2w}{dx^2}\right]_{x=0} - K_{RA}\left[\frac{dw}{dx}\right]_{x=0} = 0 \quad \text{at } x = 0 \tag{2b}$$

$$\left[\frac{d}{dx}\left(EI(x)\frac{d^2w}{dx^2}\right)\right]_{x=L} + P\left[\frac{dw}{dx}\right]_{x=L} - K_{LB}w(L) = 0 \quad \text{at } x = L \tag{2c}$$

$$EI(x)\left[\frac{d^2w}{dx^2}\right]_{x=L} + K_{RB}\left[\frac{dw}{dx}\right]_{x=L} = 0 \quad \text{at } x = L \tag{2d}$$

while the condition at point $x = qL$ is that w is completely continuous. Therefore, the deflection, slope, bending moment and shear force for the left part and right part at D should be the same.

Next, we discretize the foregoing continuous equations using the first order central finite difference method. Before the discretization, the system characteristics may be expressed as

$$EI(x) = EI_0 f(x), \quad \rho(x) = \rho_0 r(x), \quad k(x) = k_0\kappa(x) \tag{3a}$$

$$EI_j = EI_0 f_j, \quad \rho_j = \rho_0 r_j, \quad k_j = k_0\kappa_j \tag{3b}$$

where EI_j, ρ_j and k_j are the values of $EI(x)$, $\rho(x)$ and $k(x)$ at node j, respectively. Therefore, it should be noted that $EI_j = EI(x = x_j)$, $\rho_j = \rho(x = x_j)$ and $k_j = k(x = x_j)$. Besides, for the special case of $j = 0$, f_0, r_0 and κ_0 are equal to 1.

The governing equation (1) is hereby transformed to the following difference equations using the first order central finite difference method:

$$f_{j+1}w^{(1)}_{j+2} + \left(\frac{\alpha}{n^2} - 2f_j - 2f_{j+1}\right)w^{(1)}_{j+1} + \left(f_{j+1} + 4f_j + f_{j-1} - \frac{2\alpha}{n^2} - \frac{\Omega^2}{n^4}r_j + \frac{\xi}{n^4}\kappa_j\right) w^{(1)}_j + \left(\frac{\alpha}{n^2} - 2f_j - 2f_{j-1}\right)w^{(1)}_{j-1} + f_{j-1}w^{(1)}_{j-2} = 0 \quad \text{for } j = 0\ldots i \tag{4a}$$

$$f_{j+1}w^{(2)}_{j+2} + \left(\frac{\alpha}{n^2} - 2f_j - 2f_{j+1}\right)w^{(2)}_{j+1} + \left(f_{j+1} + 4f_j + f_{j-1} - \frac{2\alpha}{n^2} - \frac{\Omega^2}{n^4}r_j\right) w^{(2)}_j + \left(\frac{\alpha}{n^2} - 2f_j - 2f_{j-1}\right)w^{(2)}_{j-1} + f_{j-1}w^{(2)}_{j-2} = 0 \quad \text{for } j = i\ldots n \tag{4b}$$