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A state space solution for onset of surface instability of elastic cylinders with radially graded Young's modulus

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ABSTRACT

A state space solution is developed to analyze surface instability of cylindrical structures with Young's modulus varying arbitrarily in the radial direction. By using the incremental theory for surface instability of elastic materials, the equilibrium equations for the incremental stress field from a fundamental state are derived for radially graded elastic cylinders subjected to an axial compression, which together with the boundary conditions constitute an eigenvalue problem. In the present work, a state space method is established to solve the eigenvalue problem and predict the critical condition for onset of surface instability. The state space solutions for three typical examples are presented and shown to be in good agreement with the numerical results by the finite element method, including the analytical solution for a thin cylindrical shell. In particular, a transition of the critical buckling mode for a soft cylinder covered by a bilayer is illustrated clearly by the present method. In contrast to the finite element method, the state space method is a semi-analytical approach with higher computational efficiency for arbitrarily graded elastic cylinders, including layered structures.

1. Introduction

Graded cylindrical structures widely exist in nature and engineering applications. On one hand, some biological organs, such as plant stems, human or animal blood vessels, and earthworm bodies, can be seen as elastic cylindrical structures with material properties varying in the radial direction, in which the stiffer layer may provide effective support and/or protection for the other softer tissues. On the other hand, in many engineering structures, functionally graded cylinders, thin-walled cylindrical shells filled with soft cores, and other similar components are often used in aerospace, nuclear reactor, chemical plant, and civil engineering (Karam and Gibson, 1995a; Ye et al., 2011). Such components have advantages of corrosion resistance, high temperature resistance and/or light weight. When subjected to axial compression, the compressive stress inside the system may cause surface instability. For a cylinder used for bearing axial pressure, surface instability in general needs to be avoided. However, this phenomenon of surface instability has recently been exploited for a range of applications, e.g., sensors (Schaffer et al., 2000; Stafford et al., 2004), microfluidic devices (Beebe et al., 2000; Sugiura et al., 2007), micro-optics (Harrison et al., 2004), active surfaces (Tokarev and Minko, 2009), and soft electronics and actuators (Yang et al., 2010; Rogers et al., 2010). Therefore, whether to use or to avoid surface instability, understanding the intrinsic mechanism of this phenomenon is of great importance.

Creases and wrinkles are two types of surface instability patterns. The creasing instability often initiates on a homogeneous block of rubberlike elastic material with a large compressive strain (Biot, 1963; Gent and Cho, 1999; Tallinen et al., 2013); while wrinkling instability readily occurs for a stiff skin on a compliant substrate, the critical strain can be much smaller (Huang et al., 2005; Cao and Hutchinson, 2012), suitable for linear elastic analysis. For a graded elastic layer with stiffness decaying from the surface to the interior, wrinkling instability may also happen on the surface with a very small in-plane compressive strain (Lee et al., 2008; Wu et al., 2014). In the past decades, most theoretical and experimental studies have focused on elastic materials with a planar surface (Biot, 1963; Huang et al., 2005). Although the investigation for instability on a curved surface is a more complex problem, there have been some pioneering works on surface wrinkles in cylindrical shells with or without a soft core. By using the shallow shell theory, Koiter (1945, 2009) derived a classical solution for onset of wrinkling in a thin cylindrical shell under axial compression. Later, the wrinkling instability of a cylindrical shell with a soft core under axial compression and lateral pressure was considered by means of Donnell's equations (Seide, 1962). By re-analyzing the buckling of a thin cylindrical shell with a compliant core, Karam and Gibson (1995a, 1995b) found that the buckling resistance of a

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hollow cylindrical shell could be improved significantly by infilling a compliant elastic core, predicting that there is a great potential for bio-mimicking of natural structures in engineering. The same system was recently investigated by Arani et al. (2007) using an energy method and they concluded that the application of an elastic core may increase elastic stability and significantly reduce the weight of cylindrical shells. To further determine the effect of the filled core thickness on the behavior of buckling, Ye et al. (2011) developed a simple formula by employing the Rayleigh–Ritz approximation to predict the critical buckling stress, where the core material can be stiffer or softer than the shell material. More recently, a theoretical analysis based on the nonlinear Donnell–Mushtari–Vlassov shell theory (Novozhilov, 1959) was conducted by Zhao et al. (2014) for a cylindrical shell supported by a soft core subjected to axial compression, in which the surface wrinklings along the axial and circumferential directions are both analyzed. In these theoretical analyses for core-shell cylindrical structures, the shell is often regarded to be very thin and the approximate theory associated with thin plates has been used frequently, which requires that the buckling wavelength be much larger than the thickness of the shell. If the shell is relatively thick and its material is inhomogeneous, the above approach would be no longer applicable.

In this paper, we devote our effort on the critical condition for surface instability of radially graded elastic cylinders. We note that a similar eigenvalue problem was investigated by the finite element method for an elastic half space with graded material properties (Lee et al., 2008) and graded elastic cylinders (Jia et al., 2014). A theoretical analysis was performed by Diab and Kim (2014) for a neo-Hookean elastic half space with a stiffness decaying exponentially from the surface to the bulk. Recently, a state space method was developed for surface instability of the elastic layers and hydrogel layers with arbitrarily depth-wise graded material properties (Wu et al., 2014, 2017). The state space method was found to be computationally effective for the elastic materials with a planar surface in comparison with the finite element method. In the present study, we extend the state space method for surface instability of elastic cylinders with elastic modulus varying arbitrarily in the radial direction.

2. Theory of surface instability for elastic cylinders

In this section, we briefly review the incremental theory for surface instability of elastic materials in a Cartesian coordinate system (Wu et al., 2014), and then present the governing equations under cylindrical coordinates by the coordinate transformation for the surface instability analysis of graded elastic cylinders.

Consider an elastic cylindrical structure in the stress-free state with the inner radius A and the outer radius B as shown in Fig. 1(a), where a Cartesian coordinate system is set up with the reference coordinates X_1 and X_2 at a cross-section, and X_3 along the axis of the cylinder. When the cylinder is subjected to an axial compression, the compressive stress inside the system may give rise to instability on the inner and outer lateral surfaces. Prior to surface instability, the compressed cylinder is seen as in a fundamental state (Fig. 1(b)), and the corresponding current coordinates are denoted as (x_1, x_2, x_3).

The material is assumed to be linear elastic with a quadratic strain energy function in terms of Green–Lagrange strain

$$W = \frac{1}{2} C_{IJKL} E_{IJ} E_{KL}, \quad (1)$$

where $E_{IJ} = \frac{1}{2}(F_{kI}F_{kJ} - \delta_{IJ})$, $F_{kJ} = \partial x_k / \partial X_J$, δ_{IJ} is the Kronecker delta, and C_{IJKL} is the elastic modulus. For an isotropic elastic cylinder with material properties varying in the radial direction, C_{IJKL} is a function of X_1 and X_2 and possesses the isotropic symmetry.

From the strain energy density function in Eq. (1), we obtain the second Piola–Kirchhoff stress:

$$S_{IJ} = \frac{\partial W}{\partial E_{IJ}} = C_{IJKL} E_{KL}, \quad (2)$$

and the first Piola–Kirchhoff stress (nominal stress):

$$P_{iJ} = \frac{\partial W}{\partial F_{iJ}} = F_{iK} S_{KJ}. \quad (3)$$

Under axial compression, strain compatibility requires that the imposed axial nominal strain $\varepsilon_0 = F_{33} - 1$ is identical everywhere in the fundamental state.

Next consider an incremental displacement Δu_i $(i = 1, 2, 3)$ from the fundamental state. The increments of the deformation gradient and the Green–Lagrange strain are

$$\Delta F_{kJ} = F_{iJ} \Delta u_{k,i}, \quad (4)$$

$$\Delta E_{IJ} = \frac{1}{2}(F_{kI}F_{iJ} + F_{iI}F_{kJ}) \Delta u_{k,i}, \quad (5)$$

here and subsequently, the notation $(\ )_{,j}$ denotes differentiation with respect to x_j in the fundamental state, accordingly, $(\ )_{,J}$ is differentiation with respect to X_J in the reference state; the Einstein summation convention is implied over repeated indices unless noted otherwise. Correspondingly, the increments of the Piola–Kirchhoff stresses are

$$\Delta S_{IJ} = C_{IJKL} \Delta E_{KL}, \quad (6)$$

$$\Delta P_{iJ} = F_{iK} \Delta S_{KJ} + S_{KJ} \Delta F_{iK}. \quad (7)$$

Assuming the lateral surfaces of elastic cylindrical structures are traction free, the incremental stress field must satisfy the following equilibrium equation and boundary condition:

$$\Delta P_{iJ,J} = (F_{iK} \Delta S_{KJ} + S_{KJ} \Delta F_{iK})_{,J} = 0, \quad (8)$$

$$\Delta P_{iJ} \cdot N_J = (F_{iK} \Delta S_{KJ} + S_{KJ} \Delta F_{iK}) \cdot N_J = 0, \quad \text{at } \sqrt{X_1^2 + X_2^2} = A \text{ and } B, \quad (9)$$

where N_J represents the direction cosine of the outer normal relative to the coordinate X_J.

Assuming that the strain in the fundamental state is small so that $F_{iK} \approx \delta_{iK}$ and the reference coordinates could be replaced with the current coordinates, the increment of the first Piola–Kirchhoff stress in Eq. (7) can be approximated as

$$\Delta P_{iJ} \approx \Delta P_{ij} \approx C_{ijkl} \Delta u_{k,l} + P_{kj} \Delta u_{i,k}. \quad (10)$$

Thus the equilibrium equation (8) and the boundary condition (9) may be further expressed as

$$(C_{ijkl} \Delta u_{k,l} + P_{kj} \Delta u_{i,k})_{,j} = 0, \quad (11)$$

$$(C_{ijkl} \Delta u_{k,l} + P_{kj} \Delta u_{i,k}) \cdot n_j = 0, \quad \text{at } \sqrt{x_1^2 + x_2^2} = a \text{ and } b, \quad (12)$$

where n_j is the direction cosine of the outer normal relative to the coordinate x_j. Equations (11) and (12) are essentially identical to that in Lee et al. (2008) with the same assumptions of linear elasticity and small strain.

To analyze surface instability for an elastic cylinder with material properties varying in the radial direction, the cylindrical coordinates are employed for this system in the fundamental state as shown in Fig. 1(b) and (c) with the current coordinates r, θ, and z. For simplicity, in the present study we assume that only Young's modulus varies in the radial direction whereas Poisson's ratio is fixed as a constant, thus an axial symmetrical system is obtained with only one non-zero stress component P_{33} everywhere in the

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