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Capturing the effect of thickness on size-dependent behavior of plates with nonlocal theory

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ABSTRACT

The effective elastic properties of nano-structures are shown to be strongly size-dependent. In this paper, using a three dimensional strong nonlocal elasticity, we have presented a formulation to capture the size-dependent behavior of plate structures as a function of their thickness. This paper discusses some new aspects of employing a three dimensional nonlocal formulation for analysis of plates, namely, the confining of the nonlocal kernel in the near-boundary regions at the two surfaces of the plate. To address this aspect, we have studied two different types of nonlocal kernels, one bounded in a finite domain of the structure and the other, non-bounded. This study shows that the influence of the plate's thickness on its bending stiffness can be captured within the nonlocal elasticity framework, and this influence highly depends on the bounding of the nonlocal kernel. Particularly, for a uniformly deformed plate with a homogeneous isotropic material, using the nonlocal formulation with the bounded domain reflects the physics of the problem better.

1. Introduction

Micro and nano electro-mechanical-systems (MEMS and NEMS) play key roles in a wide variety of modern applications, including nano-mechanical sensors, actuators, and many electronic devices. The performance of these devices is based on movements and deformations of their micro/nano mechanical components, such as cantilevers, double clamped beams or plates. Obviously, the further development of these devices requires a thorough understanding and modeling of their mechanical behavior. However, devices at nano-meter scale may exhibit mechanical properties not noticed at the macro-scale. Many theoretical methods such as molecular and atomistic simulations and size-dependent continuum theories are being developed to analyze this behavior. Molecular and atomistic simulations are generally time consuming and computationally expensive. Alternatively, continuum models offer superior computational efficiency.

Classical continuum mechanics is size independent and it cannot provide a good prediction for small scales. Therefore, size-dependent continuum theories have been introduced to account for these scaling effects (Eringen, 2002; Kröner, 1967). In an attempt to account for atomistic effects, these theories embed an internal material length scale. This makes it possible to qualify the size of a structure as "large" or "small" relative to its material length scale (Eringen, 2002; Angela Pisano and Fuschi, 2003; Eringen, 1977). If "large", then these theories should converge to classical continuum theory, and, otherwise, they should reflect the size-dependence.

One of the best-known size-dependent continuum theories is non-local continuum theory, initiated in a general notation by Piola in 1846 (dell'Isola et al., 2014; Dell'Isola et al., 2016). In non-local continuum theory, a material point is influenced by the state of all points of the body. The mathematical description of this theory relies on the introduction of additional contributions in terms of "gradients" or "integrals" of the strain field in the constitutive equations. This, respectively, leads to so-called "weak" or "strong" non-local models (Di Paola et al., 2013; Engelbrecht and Braun, 1998; Silling and Lehoucq, 2010). Although both models have been found to be largely equivalent (Peerlings et al., 2001), the weak (gradient) formulation requires stronger continuity on the displacements gradients. In addition, in cases that a well-defined spatial interaction exists in the material, the strong (integral) approach is preferred, because it models the nonlocality in a more transparent way (Peerlings et al., 2001).

In strong nonlocal theories, particularly formulated by Kröner in 1967 (Kröner, 1967), and then by Eringen in 1977 (Eringen, 1977; 2002), the point-to-point relationship between stresses and strains does not hold anymore. Instead, the stress in each point is influenced by the strain of all points of the body. This influence is captured by a spacial integral over the body. The integral is weighted

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with a decaying kernel, which is designed to incorporate the long-range interaction between atoms in the continuum model. With the spacial integral, the dimensions of the body are brought into the constitutive equations, and thus, the constitutive equations will be size-dependent.

It is worth to mention here that closely related to strong nonlocal theory, the *peridynamics* theory has been developed by Silling and Lehoucq (2010). In fact, in peridynamics, instead of spatial differential operators, integration over differences of the displacement field is used to describe the existing, possibly nonlinear, forces between particles of the solid body (Silling and Lehoucq, 2010; Weckner et al., 2009). However, in contrast to the peridynamic theory, the strong nonlocal theories rely on spatial integrations. The present study mainly focuses on the commonly used strong formulation given by Eringen.

The strong nonlocal theory has been used in many studies for modeling micro- or nano-mechanical devices. In these studies, mechanical components such as thin-film elements and plate-like structures have been modeled with so-called *two-dimensional* non-local formulations, also known as "nonlocal plate theories" (Eringen, 1984; Lu et al., 2007a; 2007b). In these theories, the plate-like structures are generally modeled as a two-dimensional domain. In this way, the nonlocal contribution of the strain field in the transverse direction is ignored. Therefore, the size of a plate is only defined by its lateral dimensions, and thus, its thickness is not incorporated in its size-dependent behavior.

In plane stress problems, which are inherently two dimensional—such as the stress analysis near the crack tip in a thin plate (Eringen, 1977)— ignoring the nonlocal effects in transverse direction is within reason. Also, for structures whose thickness is much smaller than the material length-scale, such as a monolayer graphene, the non-local effect in transverse direction is in fact meaningless (Duan et al., 2007). However, modeling a plate as a two-dimensional domain and ignoring the nonlocal contribution in the transverse direction is not always valid. First of all, from a physical point of view, a nonlocal theory is supposed to incorporate the interaction between atoms in a continuum model and so its effect should exist in all directions (Picu, 2002). Second, since the thickness of a plate is significantly smaller than its lateral dimensions, the length scale at which classical elasticity breaks down appears in the transverse direction first. Moreover, in problems in which there is a uniform strain field in the tangential directions, the nonlocal stress as a function of weighted average of strain in tangential directions is simply equal to the classical stress. This means the two-dimensional formulation fails to reflect any size-dependency. In such a case, it is likely that transverse non-locality would have a more pronounced size-dependence contribution.

In this paper, we particularly investigate how the strong three dimensional nonlocal formulation can incorporate the plate thickness. Moreover, we study the effect of thickness in the predicted size dependence of the overall flexural rigidity and elastic modulus of the plate.

It is worth to note that in nonlocal elasticity, as a consequence of including contributions of integrals of the strain field in the constitutive equations, the differential order of the governing equations changes. This results in additional boundary conditions which should physically reflect the surface properties of the material/structure. The latter, however, has not been discussed rigorously in literature so far and instead, the boundaries are often avoided in the respective analyses. When a three dimensional nonlocal formulation is employed in the analysis of plates, these extra boundary conditions should be defined on the upper and lower surface of the plate. In order to investigate the significance of these boundary conditions, two different treatments of the boundaries will be addressed in this paper.

This paper is structured as follows. In Section 2, the fundamentals of Eringen's nonlocal elasticity theory, some important considerations and the basis of conventional nonlocal plate theory are reviewed. In Section 3, we will use a three dimensional nonlocal formulation to solve an example of uniformly deformed plate. For this purpose two types of boundary conditions will be applied for the nonlocal formulation. The results of this analysis will be discussed and compared to classical plate theory in Section 4. In the last section, the conclusions of this study will be presented.

2. Nonlocal elasticity theory

In linear nonlocal elasticity, the stress tensor (**t**) for a homogenous and continuous domain is determined as:

$$\begin{aligned} t_{ij}(\mathbf{x}) &= \int_{V_b} \alpha\left(|\mathbf{x}-\mathbf{x}'|, e_0 a\right) C_{ijkl}\varepsilon_{kl}(\mathbf{x}')dV(\mathbf{x}') \\ &= \int_{V_b} \alpha\left(|\mathbf{x}-\mathbf{x}'|\right)\sigma_{ij}(\mathbf{x}')dV(\mathbf{x}') \end{aligned} \tag{1}$$

where $\varepsilon_{kl}(\mathbf{x}')$ are the classical Cauchys strain components at the point $\mathbf{x}'$ and C_{ijkl} are the components of the elasticity tensor (Eringen, 2002; 1983). Index k and l are the dummy index in Einstein's summation convention, and Cartesian coordinates have been assumed. The product of these two terms can be simply substituted with classical stress component $\sigma_{ij}(\mathbf{x}')$, as in the second line. Then, V_b is the volume of the body at hand. The function $\alpha(|\mathbf{x}-\mathbf{x}'|, e_0 a)$ is the non-local kernel representing the effect of long-range interactions (Silling and Lehoucq, 2010). This radial kernel reflects the nonlocal contribution of strain in all points $\mathbf{x}'$ of the body. The nonlocal kernel α is also a function of parameters a and e_0. The parameter a is the material characteristic length scale (e.g. atomic distance, lattice parameter, granular distance) (Eringen, 1984), and e_0 is a constant for adjusting the model to match experiments or other models (Lu et al., 2007a; Eringen, 1977; Picu, 2002). Other properties of the nonlocal kernel α will be discussed later in this section.

It should be stressed that the proof of existence of Cauchys stress tensor is based on the equilibrium of contact forces with a force which is assumed to be continuous in space. We may use a similar assumption as well (as proposed in dell'Isola et al., 2014; Dell'Isola et al., 2016). Moreover, in strain gradient nonlocal theories, the constitutive equations are much more than one stress-strain relationship. Instead, so-called *double* or *hyper* stress components are defined associated to higher order strain gradients (Lam et al., 2003). In the strong nonlocal theory, however, the basic equations for an isotropic solid can be expressed in its simplest form as in Eq. (1) (Eringen, 2002; Kröner, 1967; Peerlings et al., 2001; Di Paola et al., 2013; Engelbrecht and Braun, 1998; Silling and Lehoucq, 2010).

Accordingly, the nonlocal strain energy is expressible as Eringen (1977):

$$U_{nonlocal} = \frac{1}{2}\int_{V_b} t_{ij}\varepsilon_{ij}dV. \tag{2}$$

Please note that this formulation of internal energy is a particular case of the formulation given by Kröner (1967), provided that the kernel α reflects the local (short-range) as well the nonlocal (long-range) effects. The equilibrium equations in the nonlocal continuum theory are the same as for classical continuum theory, but represented in terms of the nonlocal stresses (t_{ij}) rather than the local stresses (σ_{ij}).

2.1. Nonlocal kernel

The function used as the nonlocal kernel $(\alpha(|\mathbf{x}-\mathbf{x}'|, e_0 a))$ needs to have the following characteristic properties;