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Exact connections between the effective elastic moduli of fibre-reinforced composites with general imperfect interfaces

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Transversely isotropic composites consisting of an elastic matrix reinforced by unidirectional elastic fibres of circular cross-section are studied. The displacement and traction vectors at the interface between a fibre and the matrix are assumed to be simultaneously discontinuous and governed by a general isotropic imperfect interface model. It is first proved that piecewise uniform strain fields transversely isotropic about the fibre direction can be generated inside such a composite. The existence of these piecewise uniform strain fields is then exploited so as to derive two exact connections between the five effective elastic moduli of the composite. These two exact connections are finally shown to cover as particulars cases all the relevant results reported in the literature for fibre-reinforced composites.

1. Introduction

In a famous work (Hill, 1964), Hill proved that two exact connections exist between the five effective elastic moduli of any transversely isotropic composite consisting of a matrix reinforced by unidirectional straight fibres. The usefulness of this elegant result is at least twofold. First, it reduces the number of the independent effective moduli of such a composite from five to three. Second, it can serve as a benchmark for testing the consistency and correctness of homogenization techniques and micromechanical methods elaborated for determining the effective elastic moduli of the fibre-reinforced composites in question (see, e.g., Hashin and Rosen, 1964; Christensen and Lo, 1979; Benveniste, 1987; Bonnet, 2007).

In the aforementioned work of Hill (1964), the interface between the matrix and a fibre was assumed to be perfect in the sense that both the displacement and traction vectors are continuous across the interface. However, in a variety of situations of theoretical and/or practical interest, the perfect interface assumption fails to hold. For example, in the case of fibrous nanocomposites, the interface stress may be so important that the traction vector is no longer continuous across the interfaces (see, e.g., Chen et al., 2007; Le Quang and He, 2007; 2008; Mogilevskaya et al., 2008; 2010a; 2010b).

Some authors (see, e.g., Chen and Dvorak, 2006; Duan and Karihaloo, 2007) have extended the aforementioned result of Hill to accounting for imperfect interfaces. However, in all the extended results reported in the literature, the imperfect interface models adopted are the spring-layer model and/or the membrane-type (or Gurtin-Murdoch) one. In the former, the traction vector is continuous across an interface while the displacement vector suffers from a jump across the same interface, which is usually assumed to be proportional to the traction vector. In the latter, the displacement vector is continuous across an interface whereas the traction vector exhibits a jump across the same interface, which has to be compatible with the generalized Laplace-Young equation. It seems that, in the general case where both the displacement and traction vectors suffer jumps across an interface, no results have been reported on the connections between the effective elastic moduli of fibrous composites. The present work aims to provide such results.

In a series of papers (see, e.g., Bövik, 1994; Hashin, 2002; Benveniste, 2006; Gu and He, 2011) it has been demonstrated that both the spring-layer and membrane-type models correspond in reality to the two extreme particular cases of a general linear imperfect interface model derived by considering two physically sound configurations and requiring them to be equivalent to within a tolerable error. Precisely, the starting configuration is a three-phase one where a linearly elastic interphase of small uniform thickness h is located between, and perfectly bonded to, two linearly elastic phases. To facilitate our explanation, let us denote by S_1 and S_2 the perfect interfaces between these two phases and the interphase. The final configuration is a two-phase one where

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the interphase has been replaced by an imperfect interface of null thickness located at the middle surface S of the interphase and the two phases have been extended, respectively, from S_1 and S_2 to S. The displacement and traction jumps across the imperfect interface S in the final two-phase configuration, characterizing the general linear imperfect interface model in question, are required to be such that the displacement and traction jumps across the interphase in the starting three-phase configuration are, to within an error of order $0(h^2)$, equal to the ones across the zone bounded by the fictitious surfaces S_1 and S_2 comprising the imperfect interface S in the final two-phase configuration. More details can be found in the paper of Gu and He (2011). The general linear elastic imperfect interface model thus obtained reduces to the spring-layer or membrane-type model according as the interphase is much softer or much stiffer than the connected neighboring phases. Moreover, it is applicable to all the intermediate situations which are physically out of the scope of the spring-layer and membrane-type models.

The present work aims mainly at further extending Hill's result to fibre-reinforced composites where the matrix-fibre interfaces are imperfect and characterized by the general linear imperfect interface model described above. The exact connections derived for the effective elastic moduli include as special cases all the relevant results reported in the literature. In particular, we retrieve the results for fibrous nanocomposites by requiring the stiffness of an interphase to be much higher than the ones of the matrix and fibrous phases. It is worth emphasizing that all the parameters involved in our results bear clear physical and/or geometrical interpretations.

The following sections are organized as follows. In Section 2, the fibre-reinforced composites under consideration are specified by describing their microstructure, presenting their local and global constitutive laws and compactly formulating the general elastic isotropic imperfect interface model adopted. Section 3 is dedicated to showing that piecewise uniform strain fields can be generated inside a fibre-reinforced composite with general elastic isotropic interfaces once the parameters of a boundary loading transversely isotropic about the fibre direction satisfy some appropriate relations. In Section 4, the existence of piecewise uniform strain fields is exploited so as to establish two exact connections between three of the five effective elastic moduli of the fibre-reinforced composite under consideration. These two exact connections depend, in particular, on the size of the fibres. In Section 5, to illustrate the versatility, and also to check the correctness, of our results, we show that they reduce to the corresponding results relative to the spring-layer and membrane interface models when the interfacial material parameters are properly chosen. In Section 6, a few concluding remarks are provided.

2. Fibre-reinforced composites with general linear imperfect interfaces

Consider a composite material made of a linearly elastic isotropic matrix in which linearly elastic isotropic fibres are embedded and aligned along one direction (Fig. 1). Let us introduce a three-dimensional (3D) orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ such that the unit vector $\boldsymbol{e}_3$ is oriented along the direction of the fibres. In what follows, the fibres will be referred to as phase 1 and the matrix as phase 2. We make the assumption that the distribution of the fibres in the matrix is statistically homogeneous in any plane transverse to the fibre direction $\boldsymbol{e}_3$ and that no direct contact takes place between any two fibres. Denote by Γ_i the interface between the matrix and a generic fibre, say fibre i, and symbolize by $\boldsymbol{n}$ the unit vector normal to Γ_i and oriented from the fibre into the matrix.

By assumption, the mechanical behavior of the matrix and fibre phases is characterized by the isotropic Hooke law:

$$\boldsymbol{\sigma}^{(i)} = \mathbb{L}^{(i)}\boldsymbol{\varepsilon}^{(i)}, \quad \mathbb{L}^{(i)} = 3k^{(i)}\left(\frac{1}{3}\boldsymbol{I}\otimes\boldsymbol{I}\right) + 2\mu^{(i)}\left(\mathbb{I} - \frac{1}{3}\boldsymbol{I}\otimes\boldsymbol{I}\right). \tag{1}$$

Above, the superscript i equals 1 for the fibre phase and 2 for the matrix phase; $\boldsymbol{\sigma}^{(i)}$ corresponds to the Cauchy stress tensor of phase i which has to satisfy the equilibrium equation

$$\mathrm{div}\boldsymbol{\sigma}^{(i)} = 0$$

without body forces; $\boldsymbol{\varepsilon}^{(i)}$ is the infinitesimal strain tensor derived from the displacement field $\boldsymbol{u}^{(i)}$ of phase i by

$$\boldsymbol{\varepsilon}^{(i)} = \frac{1}{2}\left[\nabla\boldsymbol{u}^{(i)} + \left(\nabla\boldsymbol{u}^{(i)}\right)^T\right]; \tag{2}$$

$\mathbb{L}^{(i)}$ represents the elastic isotropic stiffness tensor of phase i. In the expression of $\mathbb{L}^{(i)}$, $k^{(i)}$ and $\mu^{(i)}$ are the bulk and shear moduli, $\boldsymbol{I}$ is the 3D second-order identity tensor, $\otimes$ represents the usual tensor product, and $\mathbb{I}$ symbolizes the fourth-order identity tensor for the space of second-order symmetric tensors. With the aid of the Kronecker tensor product $\underline{\overline{\otimes}}$ defined by $\left(\boldsymbol{X}\underline{\overline{\otimes}}\boldsymbol{Y}\right)_{ijkl} = \left(X_{ik}Y_{jl} + X_{il}Y_{jk}\right)/2$ for any two second-order tensors $\boldsymbol{X}$ and $\boldsymbol{Y}$, we indeed have $\mathbb{I} = \boldsymbol{I}\underline{\overline{\otimes}}\boldsymbol{I}$.

Preliminarily, we also introduce some operators relative to the interface Γ_i between fibre i and the matrix. First, the normal and tangential projection operators of second order, $\boldsymbol{N}$ and $\boldsymbol{T}$, are defined by

$$\boldsymbol{N} = \boldsymbol{I} - \boldsymbol{T} = \boldsymbol{n}\otimes\boldsymbol{n}, \tag{3}$$

while the ones of fourth order, $\mathbb{N}$ and $\mathbb{T}$, are given by

$$\mathbb{T} = \mathbb{I} - \mathbb{N} = \boldsymbol{T}\underline{\overline{\otimes}}\boldsymbol{T}. \tag{4}$$

Next, we define the interfacial jump $[[\bullet]]$, the interfacial average $\langle\bullet\rangle$ and the surface divergence $\mathrm{div}_s(\bullet)$ through

$$[[\bullet]] = \bullet^{(+)} - \bullet^{(-)}, \quad \langle\bullet\rangle = \left(\bullet^{(+)} + \bullet^{(-)}\right)/2, \tag{5}$$

$$\mathrm{div}_s(\bullet) = \nabla(\bullet) : \boldsymbol{T}. \tag{6}$$

In the above first two definitions, $\bullet^{(+)}$ and $\bullet^{(-)}$ denote a quantity $\bullet$ evaluated at the interface Γ_i but on the respective sides of the matrix and fibre.

The interface Γ_i, assumed to be imperfect, is described by the general linear elastic isotropic imperfect interface model compactly formulated in Gu et al. (2014). As explained in the introduction, this model is derived through substituting an imperfect interface of null thickness for a linearly elastic isotropic interphase, called phase 0, of small uniform thickness h perfectly bonded to the fibre and matrix. Precisely, the imperfect interface model is characterized by the following two jumps across Γ_i:

$$[[\boldsymbol{u}]] = \frac{h}{2}[c_1(\boldsymbol{T} : \langle\boldsymbol{\varepsilon}\rangle)\,\boldsymbol{n} + (c_2\boldsymbol{N} + c_3\boldsymbol{T})\langle\boldsymbol{t}\rangle]; \tag{7}$$

$$[[\boldsymbol{t}]] = -\mathrm{div}_s\boldsymbol{\sigma}_s \tag{8}$$

with

$$\boldsymbol{\sigma}_s = -\frac{h}{2}[c_1(\boldsymbol{n}\cdot\langle\boldsymbol{t}\rangle)\boldsymbol{T} + (c_4\mathbb{T} + c_5\boldsymbol{T}\otimes\boldsymbol{T})\langle\boldsymbol{\varepsilon}\rangle]. \tag{9}$$

In the above expression, $\boldsymbol{\sigma}_s$ represents the interfacial (or membrane) stress tensor which is two-dimensional and tangent to Γ_i, $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$ corresponds to the traction vector acting on the fibre or matrix side of Γ_i, and c_j $(j = 1, \ldots, 5)$ are the interfacial material parameters determined by the bulk and shear moduli, $k^{(i)}$ and $\mu^{(i)}$ $(i = 0,\ 1,\ 2)$, of the interphase, fibre and matrix phases:

$$c_1 = \frac{l'^{(2)}}{n'^{(2)}} + \frac{l'^{(1)}}{n'^{(1)}} - 2\frac{l'^{(0)}}{n'^{(0)}}, \quad c_2 = \frac{2}{n'^{(0)}} - \frac{1}{n'^{(2)}} - \frac{1}{n'^{(1)}},$$