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Rigid-plastic torsion of a hollow tube in strain-gradient plasticity

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ABSTRACT

Closed-form expressions for the Cauchy stress, microstress, moment-stress, and the torque-twist relationship in a twisted hollow circular tube are derived for a rigid-plastic strain-gradient plasticity. This is accomplished for any of the gradient-enhanced effective plastic strain measures from a considered broad class of these measures. Numerical results are given and discussed for the two most frequently utilized measures and for the three adopted stress-strain relationships modeling the uniaxial tension test. Solid circular rods and thin-walled tubes are both considered. The existence of the line forces is also discussed from the standpoint of the basic equilibrium considerations and the principle of virtual work.

1. Introduction

In the wake of Fleck and Hutchinson (1993) and Fleck et al. (1994) analyzes of the effects of strain-gradient on plastic response of materials at micron scale, there has been a great amount of research devoted to the development of what is now known as the strain-gradient plasticity. The representative references include, inter alia, the contributions by Fleck and Hutchinson (1997, 2001); Nix and Gao (1998); Gao et al. (1999); Huang et al. (2000, 2004); Hutchinson (2000, 2012); Gurtin (2002, 2003, 2004); Gudmundson (2004); Anand et al. (2005a); Gurtin and Anand (2005a, 2005b, 2009); Bardella (2006, 2007); Fleck and Willis (2009a, 2009b); Polizzotto (2009); Voyiadjis et al. (2010); Dahlberg et al. (2013); Nielsen and Niordson (2014); Fleck et al. (2014, 2015); Bardella and Panteghini (2015); Anand et al. (2015b). In most of these works the material length parameter is introduced in the theory through the definition of the gradient-enhanced effective plastic strain, which combines the contributions from the effective plastic strain and the effective plastic strain-gradient. Physically, the size-dependence in non-uniform deformation problems at micron scale has been attributed to the existence of large gradients of plastic strain and the associated network of the so-called geometrically necessary dislocations. The increase of the plastic collapse limit load with the decreasing specimen size was pointed out and elaborated upon by Polizzotto (2010, 2011).

The objective of the present paper is to derive the complete stress field (microstress, moment-stress, and the Cauchy stress) and the torque-twist relationship for a twisted hollow circular tube made of a rigid-plastic material by using any of the gradient-enhanced effective plastic strain measures from a wide class of these measures frequently adopted in the literature. Special attention is given to measures defined by the linear and harmonic sum of the effective plastic strain and its gradient, scaled by the material length. The torsion testing of thin wires was a benchmark problem demonstrating the size effect at micron scale (Fleck et al., 1994), which was further studied, using different constitutive models, by many investigators, including (Fleck and Hutchinson, 1997; Fleck et al., 2014; Huang et al., 2000; Gudmundson, 2004; Voyiadjis and Abu Al-Rub, 2005; Idiart et al., 2009; Polizzotto, 2011; Liu et al., 2013; Lubarda, 2016a). The general results derived in this paper hold for an arbitrary expression representing the stress-plastic strain response in simple tension, although numerical evaluations are performed by adopting three specific expressions.

2. Gradient-enhanced effective plastic strain

In a simple formulation of the deformation theory of strain-gradient plasticity (Hutchinson, 2012), the specific plastic work (per unit volume) is expressed in terms of the gradient-enhanced effective plastic strain E_p by

$$w_p(E_p) = \int_0^{E_p} \sigma_0(\epsilon_p)\, d\epsilon_p\,, \tag{1}$$

where $\sigma_0 = \sigma_0(\epsilon_p)$ represents the stress-strain curve from the uniaxial tension test. The expression (1) implies that the plastic work needed to deform the material element in the presence of strain-gradients is equal to that at the same strain in the absence of gradients.

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A wide class of the gradient-enhanced effective plastic strain measures, each involving one material length parameter l, is (Evans and Hutchinson, 2009; Fleck and Hutchinson, 1997)

$$E_{\rm p} = \left(e_{\rm p}^s + l^s g_{\rm p}^s\right)^{1/s}, \quad (s \geq 1), \tag{2}$$

where $e_{\rm p}$ is the effective plastic strain and $g_{\rm p}$ the effective plastic strain-gradient, defined by

$$e_{\rm p} = \left(\frac{2}{3}\epsilon_{ij}^{\rm p}\epsilon_{ij}^{\rm p}\right)^{1/2}, \quad g_{\rm p} = \left(\frac{2}{3}\epsilon_{ij,k}^{\rm p}\epsilon_{ij,k}^{\rm p}\right)^{1/2}. \tag{3}$$

The two most frequently used measures are associated with the choices $s = 1$ and $s = 2$, which specify $E_{\rm p}$ as either a linear or harmonic sum of $e_{\rm p}$ and $lg_{\rm p}$,

$$E_{\rm p} = e_{\rm p} + lg_{\rm p}, \quad E_{\rm p} = \left(e_{\rm p}^2 + l^2 g_{\rm p}^2\right)^{1/2}. \tag{4}$$

In general, when fitting experimental data, a different value of l may be needed for each choice of s. The choice $s = 2$ is particulary appealing from the mathematical point of view and was used in most studies of the strain-gradient plasticity, although in some studies the choice $s = 1$ was found to be more attractive (Evans and Hutchinson, 2009).

The plastic strain is taken to be

$$\epsilon_{ij}^{\rm p} = e_{\rm p} m_{ij}, \quad m_{ij} = \frac{3}{2}\frac{\sigma'_{ij}}{\sigma_{\rm eq}}, \tag{5}$$

where the equivalent stress is

$$\sigma_{\rm eq} = \left(\frac{3}{2}\sigma'_{ij}\sigma'_{ij}\right)^{1/2}, \tag{6}$$

with the prime designating a deviatoric part. The total infinitesimal strain is

$$\epsilon_{ij} = \epsilon_{ij}^{\rm e} + \epsilon_{ij}^{\rm p}, \tag{7}$$

with the elastic component related to the Cauchy stress by the isotropic generalized Hooke's law

$$\epsilon_{ij}^{\rm e} = \frac{1}{2\mu}\sigma'_{ij} + \frac{1}{9\kappa}\sigma_{kk}\delta_{ij}, \quad \sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}. \tag{8}$$

The shear and bulk moduli are μ and κ.

3. Work-conjugates to plastic strain and its gradient

It is assumed that plastic strain-gradients $\epsilon_{ij,k}^{\rm p}$ contribute to the work per unit volume. The work conjugate to $\epsilon_{ij,k}^{\rm p}$ is the moment-stress $\tau_{ijk} = \tau_{jik}$. The work-conjugate to $\epsilon_{ij}^{\rm p}$ is the microstress $q_{ij} = q_{ji}$, such that (Gudmundson, 2004)

$$\dot{w}_{\rm p} = q_{ij}\dot{\epsilon}_{ij}^{\rm p} + \tau_{ijk}\dot{\epsilon}_{ij,k}^{\rm p}. \tag{9}$$

To identify q_{ij} and τ_{ijk}, we reconcile (9) with the rate of plastic work expression

$$\dot{w}_{\rm p} = \sigma_0(E_{\rm p})\dot{E}_{\rm p}, \tag{10}$$

following from (1). Since, by the differentiation of (2) and (3),

$$\dot{E}_{\rm p} = E_{\rm p}^{1-s}\left(e_{\rm p}^{s-1}\dot{e}_{\rm p} + l^s g_{\rm p}^{s-1}\dot{g}_{\rm p}\right), \tag{11}$$

and

$$\dot{e}_{\rm p} = \frac{2}{3e_{\rm p}}\epsilon_{ij}^{\rm p}\dot{\epsilon}_{ij}^{\rm p}, \quad \dot{g}_{\rm p} = \frac{2}{3g_{\rm p}}\epsilon_{ij,k}^{\rm p}\dot{\epsilon}_{ij,k}^{\rm p}, \tag{12}$$

we obtain

$$\dot{E}_{\rm p} = \frac{2}{3}E_{\rm p}^{1-s}\left(e_{\rm p}^{s-2}\epsilon_{ij}^{\rm p}\dot{\epsilon}_{ij}^{\rm p} + l^s g_{\rm p}^{s-2}\epsilon_{ij,k}^{\rm p}\dot{\epsilon}_{ij,k}^{\rm p}\right). \tag{13}$$

The substitution of (13) into (10) and the comparison with (9) establishes, up to their workless terms, the work-conjugates

$$q_{ij} = \frac{2}{3}\frac{\sigma_0(E_{\rm p})}{E_{\rm p}^{s-1}}e_{\rm p}^{s-2}\epsilon_{ij}^{\rm p}, \quad \tau_{ijk} = \frac{2}{3}l^s\frac{\sigma_0(E_{\rm p})}{E_{\rm p}^{s-1}}g_{\rm p}^{s-2}\epsilon_{ij,k}^{\rm p}. \tag{14}$$

Clearly, $q_{ii} = 0$ and $\tau_{iik} = 0$, because $\epsilon_{ii}^{\rm p} = 0$. Expressions (14) can also be deduced directly from $q_{ij} = \partial w_{\rm p}/\partial\epsilon_{ij}^{\rm p} = \sigma_0\partial E_{\rm p}/\partial\epsilon_{ij}^{\rm p}$ and $\tau_{ijk} = \partial w_{\rm p}/\partial\epsilon_{ij,k}^{\rm p} = \sigma_0\partial E_{\rm p}/\partial\epsilon_{ij,k}^{\rm p}$, as done by Liu et al. (2013).

In particular, for $s = 1$, the microstress and the moment-stress are

$$q_{ij} = \frac{2}{3}\frac{\sigma_0(E_{\rm p})}{e_{\rm p}}\epsilon_{ij}^{\rm p}, \quad \tau_{ijk} = \frac{2}{3}l\frac{\sigma_0(E_{\rm p})}{g_{\rm p}}\epsilon_{ij,k}^{\rm p}, \tag{15}$$

while for $s = 2$,

$$q_{ij} = \frac{2}{3}\frac{\sigma_0(E_{\rm p})}{E_{\rm p}}\epsilon_{ij}^{\rm p}, \quad \tau_{ijk} = \frac{2}{3}l\frac{\sigma_0(E_{\rm p})}{E_{\rm p}}\epsilon_{ij,k}^{\rm p}. \tag{16}$$

4. Principle of virtual work

The principle of virtual work of strain-gradient plasticity reads (Gudmundson, 2004; Gurtin and Anand, 2005a, 2005b; Fleck et al., 2014)

$$\int_V (\sigma_{ij}\delta\epsilon_{ij}^{\rm e} + q_{ij}\delta\epsilon_{ij}^{\rm p} + \tau_{ijk}\delta\epsilon_{ij,k}^{\rm p}){\rm d}V = \int_S (T_i\delta u_i + t_{ij}\delta\epsilon_{ij}^{\rm p}){\rm d}S, \tag{17}$$

provided that the equations of equilibrium hold

$$\sigma_{ij,j} = 0, \quad \tau_{ijk,k} + \sigma'_{ij} - q_{ij} = 0, \tag{18}$$

and the traction-stress relations

$$T_i = \sigma_{ij}n_j, \quad t_{ij} = \tau_{ijk}n_k \tag{19}$$

between the traction vector T_i and the Cauchy stress tensor σ_{ij}, and between the (deviatoric) moment-traction tensor t_{ij} are the moment-stress tensor τ_{ijk}. The components of the outward unit vector, orthogonal to the considered surface element, are denoted by n_i. The displacement components are u_i.

In the case of a rigid-plastic material, the principle of virtual work reads

$$\int_V \left(q'_{ij}\delta\epsilon_{ij}^{\rm p} + \tau'_{ijk}\delta\epsilon_{ij,k}^{\rm p} + \frac{1}{3}\sigma_{ii}\delta\epsilon_{jj}^{\rm p}\right){\rm d}V = \int_S \left[\hat{T}_i\delta u_i + \hat{R}_i D(\delta u_i)\right]{\rm d}S. \tag{20}$$

The three independent traction components $\hat{T}_i$ are

$$\hat{T}_i = \bar{T}_i - n_i n_j R_j(D_k n_k) - D_i(n_j R_j), \tag{21}$$

with

$$\bar{T}_i = T_i + R_i(D_j n_j) - D_j t_{ij}, \quad T_i = \sigma_{ij}n_j, \tag{22}$$

while the two independent higher-order traction components $\hat{R}_i$, tangential to S, are

$$\hat{R}_i = R_i - n_i n_j R_j, \quad R_i = t_{ij}n_j, \tag{23}$$

with $t_{ij} = \tau'_{ijk}n_j n_k$. The utilized surface gradient operator is defined by $D_i = (\partial/\partial x_i) - n_i D$, where D is the projection of the gradient operator to the surface normal, $D = n_j(\partial/\partial x_j)$. The spherical component of Cauchy stress $\sigma_{ii}/3$ was used in (20) as the Lagrange multiplier, associated with the incompressibility constraint $\epsilon_{jj}^{\rm p} = 0$ (Fleck and Willis, 2009b).

If the surface S has edges, an additional term appears on the right-hand side of (20), given by

$$\sum_n \oint_{C_n} p_i\delta u_i\,{\rm d}C_n, \tag{24}$$

where p_i are the line forces along the edges C_n of the smooth parts S_n of a piece-wise smooth surface S. For example, the line force along an edge formed by the intersection of two smooth surface segments $S^{(1)}$ and $S^{(2)}$ is

$$p_i = \left[\tau'_{ijk}k_j^{(1)}n_k^{(1)} - k_i^{(1)}\tau'_{jkl}n_j^{(1)}n_k^{(1)}n_l^{(1)}\right] + \left[\tau'_{ijk}k_j^{(2)}n_k^{(2)} - k_i^{(2)}\tau'_{jkl}n_j^{(2)}n_k^{(2)}n_l^{(2)}\right], \tag{25}$$