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An exact solution for the problem of flexure of a composite beam with preliminarily strained layers under large strains. Part 2. Solution for different types of incompressible materials

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ABSTRACT

The exact solution of the problem of flexure of nonlinear-elastic composite beam with preliminarily strained layers is analyzed for large strains. The solution is obtained using the theory of superposition of large strains. Numerical results are shown for the Bartenev–Khazanovich material and for Biderman material. Nontrivial effects caused by nonlinearities are discussed.

1. Introduction

In the first part of this paper (Levin et al., 2015a) the exact analytical solution for the problem of flexure of a composite beam with preliminarily strained layers is proposed. This solution is universal in the class of isotropic, incompressible, nonlinear-elastic materials. The solution is obtained using the theory of superposition of large strains (Levin, 1988, 1998, 1999). The mathematical statement of the problem is written in coordinates of the intermediate state. The general approach to solution is proposed. It is described how to use the first integrals of the nonlinear theory of elasticity for solution; this method does not require the computation of quadratures. The numerical results are given for three-layered beams and five-layered beams made of the neo-Hookean material. The dependencies of the flexural moment on the parameter describing the rotation angle of a given transverse cross-section of the bent beam are plotted for different values of principal initial stretch. These dependencies are monotone and tend asymptotically to finite limiting values as the curvature tends to infinity. The dependencies of the parameter describing the rotation angle of a given transverse cross-section of the beam on the principal initial stretch are given for the case in which the flexural moment is not applied to the beam. It is discovered that these dependencies are not monotone. This is a nontrivial effect, which can be interpreted as the existence of a limiting rotation angle of a given transverse cross-section of the unloaded beam with preliminarily strained layers.

It is interesting to find out the specific features of the obtained solution for different constitutive equations, and to analyze the effects of physical non-linearity.

In the present part of the paper the results for the Bartenev–Khazanovich (Varga) material and for the Biderman material are given. The specific features of the asymptotic behavior of the obtained solution are investigated under large strains.

Note that these materials are incompressible. The incompressibility constraint simplifies the analytical solution significantly. The solution for incompressible, isotropic, nonlinear-elastic materials is based on a known universal solution (Rivlin, 1949a, b; Truesdell, 1972; Lurie, 1990). For compressible, isotropic, nonlinear-elastic materials there are no universal solutions, with the except for homogeneous strain (Ericksen, 1955; Truesdell, 1972). For a particular case of harmonic materials (John, 1966) it is possible to find a solution (Levin et al., 2015b). The approach for solution in this case is briefly described in the end of Section 6.

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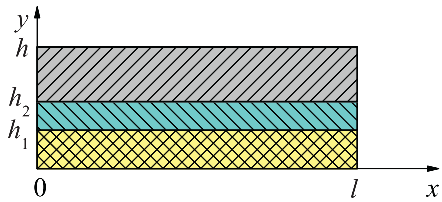

Fig. 1. The shape of a composite beam in the intermediate state.

Let us recall the mechanical problem statement. Parallelepiped-shaped beams made of incompressible materials in the unstrained (initial) state are subjected to initial strain (tension or compression) and are passed to the intermediate state. Although each beam has a homogeneous strain, it is different for each one. These beams are then joined (Fig. 1). Ideal contact conditions hold over the joint surfaces. The moment of flexure is applied to the ends of the beam. After the junction, the composite beam is deformed as a whole and the flexure of the beam occurs due to the bending moment and the initial deformation of its layers. The beam passes to the final state.

The results of solution of this problem may be used for modeling of additive manufacturing processes, technologies for production of thin film composites, coated parts using chemical vapor deposition (Pierson, 1999), spraying, atomic layer deposition, ion implantation. Stress and deformation of prestrained multilayer actuators made of dielectric elastomers (Brochu, 2012) can be analyzed using the proposed approach. The results may also be used in biomechanics, for example, to analyze the stresses in growing biological membranes (Rausch and Kuhl, 2013). And finally, the obtained solution is useful for the verification of a software for numerical solution of problems in which superposition of large strains takes place (Levin et al., 2013).

2. Nomenclature

We use the notation that is typical for the theory of superimposed large strains (Levin, 1998):

$\overset{k}{\mathbf{r}}$ — the position vector of a particle in the kth state

$\overset{k}{\nabla}$ — a gradient operator in coordinates of the kth state

$\mathbf{F}_{k,n} = \overset{k}{\nabla}\overset{n}{\mathbf{r}}$ — the deformation gradient in transition from the kth state to the nth state

$\mathbf{C}_{k,n}$ — a tensor defining the strains associated with the transition of a body from the kth state to the nth state (this tensor corresponds to Green's deformation tensor)

$\boldsymbol{\sigma}$ — the true stress tensor for the final state (the Cauchy stress tensor)

$\overset{k}{\mathbf{P}}$ — the first (nonsymmetric) Piola–Kirchhoff stress tensor in the base of the kth state

W — the strain energy density function

$\mathbf{E}$ — the identity tensor.

I_1 and I_2 — the strain invariants (Lurie, 1990)

χ_1 and χ_2 — the material response functions

p — the Lagrange multiplier

$\mathbf{f}$ — the force resultant vector

$\mathbf{m}$ — the moment resultant vector

The dot is a sign of a tensor contraction and the superscript T is a sign of transposition.

The colon is a sign of a double tensor contraction (for arbitrary second-rank tensors $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} : \mathbf{B} = A_{mn}B_{nm}$).

The initial deformation gradient $\mathbf{F}_{0,1}$ is further denoted as $\mathbf{F}_{\text{init}}$, and the additional deformation gradient $\mathbf{F}_{1,2}$ is further denoted as $\mathbf{F}_{\text{add}}$.

For brevity, the subscripts for the tensors describing the transition from the initial state to the final state are omitted; thus for example, $\mathbf{F}_{0,2} = \mathbf{F}$, $\mathbf{C}_{0,2} = \mathbf{C}$.

3. Mathematical statement of problem

Three states (configurations) of the body are distinguished: the initial (undeformed) state; the intermediate state in which the body passes after the initial strain; final state in which the body passes after the junction of layers and the application of a bending moment. These states are numbered by indices 0, 1, and 2, respectively (Levin, 1998; 1999).

The statement of the problem is written in coordinates of the intermediate state and includes the following equations.

The equilibrium equation (without body forces) (Ciarlet, 1988; Truesdell and Noll, 2013; Lurie, 1990; Levin, 1998):

$$\overset{1}{\nabla} \cdot \overset{1}{\mathbf{P}} = 0. \tag{1}$$

The relation between different stress tensors:

$$\overset{1}{\mathbf{P}} = (\det \mathbf{F}_{\text{add}})\boldsymbol{\sigma} \cdot \left(\mathbf{F}_{\text{add}}^T\right)^{-1}. \tag{2}$$

The strain energy density function W and constitutive equations for incompressible, isotropic, nonlinear-elastic materials (Ogden, 1984; Lurie, 1990):

$$W = W(I_1, I_2), \tag{3}$$

$$\overset{1}{\mathbf{P}} = [\chi_1(I_1, I_2) + I_1\chi_2(I_1, I_2)]\mathbf{F} \cdot \mathbf{F}_{\text{init}}^T - \chi_2(I_1, I_2)\mathbf{F} \cdot \mathbf{C} \cdot \mathbf{F}_{\text{init}}^T - p\left(\mathbf{F}_{\text{add}}^T\right)^{-1}, \tag{4}$$

$$\chi_1 = 2\frac{\partial W}{\partial I_1}, \quad \chi_2 = 2\frac{\partial W}{\partial I_2}, \quad I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \operatorname{tr}(\mathbf{C}^{-1}). \tag{5}$$

The incompressibility constraints are

$$\det \mathbf{F}_{\text{init}} = \det \mathbf{F}_{\text{add}} = \det \mathbf{F} = 1. \tag{6}$$

Kinematic equations (Ogden, 1984; Areias et al., 2013; Lurie, 1990; Levin, 1998):

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}, \quad \mathbf{F} = \mathbf{F}_{\text{add}} \cdot \mathbf{F}_{\text{init}}. \tag{7}$$

4. The initial deformation of the beam layers

Consider the plane strain of a rectangular composite beam occupying the region $0 \le x \le l$, $0 \le y \le h$ in the reference configuration (in the intermediate state) (Fig. 1). The Cartesian coordinates of a particle in the intermediate state are x, y, and z. The size on the z-axis is the same for all layers and is not significant. The beam consists on n layers; the kth layer occupies the region $h_{k-1} \le y \le h_k$ in the intermediate state ($0 = h_0 \le h_1 \le \cdots \le h_{n-1} \le h_n = h$). The initial (preliminary) strain of the kth layer ($k = 1, \ldots, n$) is homogeneous and given by the following deformation gradient:

$$\mathbf{F}_{\text{init}} = \lambda_{(k)}\,\mathbf{i}_1 \otimes \mathbf{i}_1 + \lambda_{(k)}^{-1}\,\mathbf{i}_2 \otimes \mathbf{i}_2 + \mathbf{i}_3 \otimes \mathbf{i}_3, \quad \lambda_{(k)} = \text{const}. \tag{8}$$

Here $\mathbf{i}_j$ are coordinate unit vectors.

If the kth layer of the beam is not prestressed, then the intermediate state of this layer coincides with the initial state, $\lambda_{(k)} = 1$, and $\mathbf{F}_{\text{init}} = \mathbf{E}$.

The materials of the composite-beam layers may be different. The strain energy density function of the kth layer ($h_{k-1} \le y < h_k$) is denoted as $W^{(k)}$. The material response functions of this layer are denoted as $\chi_1^{(k)}$ and $\chi_2^{(k)}$: $\chi_1^{(k)} = 2\frac{\partial W^{(k)}}{\partial I_1}$, $\chi_2^{(k)} = 2\frac{\partial W^{(k)}}{\partial I_2}$.

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