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Sphere rolling on a moving surface: Application of the fundamental equation of constrained motion

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ABSTRACT

This paper deals with the general formulation of the problem of a rigid sphere rolling under gravity on an arbitrarily prescribed surface that is moving in an arbitrarily prescribed manner. This is accomplished by using a recently developed modeling paradigm, which is encapsulated in a systematic general three-step procedure. The first step develops the equations of motion of the so-called unconstrained system in which the sphere is decoupled from the surface on which it moves. The novelty in this paper is the inclusion of a zero-mass particle and its associated coordinates in the unconstrained description of the system, whose equations of are trivial to write down since it is assumed that all the coordinates are independent of one another. However, this leads to a singular mass matrix. The second step involves the statement of the constraints that (a) cause the sphere to roll on the surface without slip, (b) cause the zero-mass particle to bind to the surface and to become the point of contact between the sphere and the surface, and (c) ensure that the quaternion describing the rotational motion of the sphere is a unit quaternion. The third step involves the direct application of the Udwadia–Phohomsiri equation that generates the equations of motion for the system. Simulations of the motion of a sphere rolling on a moving parabolic surface are shown illustrating the ease and efficacy with which both the formulation and the numerical results can be obtained.

The systematic modeling procedure used here to study the dynamics of the rolling sphere along with the use of a zero-mass particle opens up new ways for modeling and simulating the dynamical behavior of complex multi-body systems.

1. Introduction

The dynamics of a rigid ball rolling under gravity without slipping on a surface is one of the classical problems of mechanics in which the non-holonomic constraints play an important role, and in which the standard Lagrangian formalism in difficult to apply to readily simulate the dynamical behavior.

One of the first contributions to this problem was published by Lindelöf [1], in which it seemed that the author had completely solved the problem. Some years later, Chaplygin analyzed Lindelöf's paper and found an error. He carried out an investigation of the problem of the motion of a heavy body of revolution on a surface for a number of particular cases [2,3]. In this work Chaplygin derives the integrals of motion of the system. Despite these contributions by Chaplygin, the motion of a ball was practically unstudied until Kilin [4], derived the equations of the motion for a sphere on a plane in

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an inertial frame thereby studying the trajectories of the point of contact between the sphere and the plane. Borisov and Mamaev [5] extend this to a rigid body rolling on a plane and a sphere.

The present article presents a general formulation for obtaining the equations of motion (in an inertial frame) for a rigid sphere rolling on an arbitrarily prescribed moving surface, without slipping. The formulation allows the equations of motion to be easily and efficaciously determined by employing a new way in which the system is modeled. The aim is to develop a formulation that simplifies the modeling effort, while yielding accurate simulations of the dynamics.

The key idea underlying this new approach is the use of a suitable unconstrained system, which then is constrained appropriately to yield the system of interest—in this case, the sphere rolling on a moving surface without slipping. To accomplish this we use the approach of considering a particle of zero mass in our description of the unconstrained system.

We therefore begin with an unconstrained rigid sphere moving in an inertial coordinate frame under the force of gravity, and use an additional zero-mass particle, thereby enrolling more than the minimum number of coordinates needed to specify the configuration of the system [6]. Furthermore, in order to express the rotation of the sphere without encountering singularities in our formulation, we use quaternions, thereby increasing the number of coordinates used to describe the system even further. The equations of motion for this unconstrained system are indeed *trivial* to write down. The zero-mass particle is specifically included in our unconstrained system because it simplifies the derivation of the constraints. These constraints are applied to our unconstrained system so that: (1) the sphere rolls without slipping on the given, moving surface; (2) the zero-mass particle at each instant of time is fixed to the point of contact between the sphere and the surface, so that its motion in time represents the path traced out on the surface by the moving sphere; and, (3) the quaternion four-component column vector represents a physical rotation, and therefore has unit norm. Having described the unconstrained system (in which all the coordinates are considered independent) as above, and the appropriate constraints, the last step in the modeling procedure is the use of the Udwadia–Phohomsiri fundamental equation [7–9] which yields the required equations of motion of the constrained system.

The facility with which these equations are obtained is notable since one has only to provide the equations of motion of the unconstrained system and the constraints. The fundamental equation then automatically generates the correct equations of motion. Several numerical simulations are presented. They show that, despite the simplicity of the chosen geometries, the trajectories of the sphere have interesting non-trivial behavior, which depends in a complex manner on the motion of the surface on which it rolls.

2. The model

Consider an inertial frame of reference Oxyz in which a rigid and homogeneous sphere of radius R and mass m (with center of mass is at C) moves on an arbitrarily prescribed surface (see Fig. 1). The coordinates of C in this inertial frame are (x_C, y_C, z_C), and the surface Γ on which the sphere moves is described by the equation $f(x, y, z, t) = 0$, where the function f is assumed to be a C^2 function of it arguments. We shall also assume that the sphere and the surface are in contact only at one point W with coordinates (ξ, η, ζ), and that the sphere rolls on the surface without slip. Gravity is acting in the negative Z-direction, as shown.

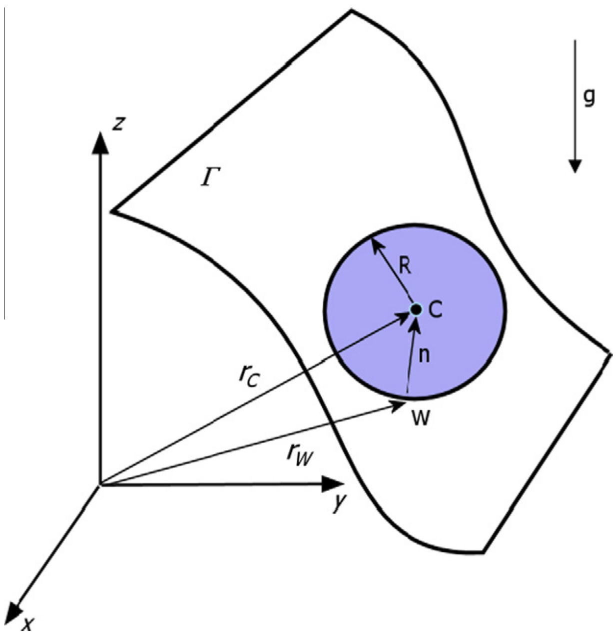

Fig. 1. Rigid sphere of radius R rolling without slipping on the moving surface Γ.