Analysis of the power flow in nonlinear oscillators driven by random excitation using the first Wiener kernel

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ABSTRACT

Random excitation of mechanical systems occurs in a wide variety of structures and, in some applications, calculation of the power dissipated by such a system will be of interest. In this paper, using the Wiener series, a general methodology is developed for calculating the power dissipated by a general nonlinear multi-degree-of freedom oscillatory system excited by random Gaussian base motion of any spectrum. The Wiener series method is most commonly applied to systems with white noise inputs, but can be extended to encompass a general non-white input. From the extended series a simple expression for the power dissipated can be derived in terms of the first term, or kernel, of the series and the spectrum of the input. Calculation of the first kernel can be performed either via numerical simulations or from experimental data and a useful property of the kernel, namely that the integral over its frequency domain representation is proportional to the oscillating mass, is derived. The resulting equations offer a simple conceptual analysis of the power flow in nonlinear randomly excited systems and hence assist the design of any system where power dissipation is a consideration. The results are validated both numerically and experimentally using a base-excited cantilever beam with a nonlinear restoring force produced by magnets.

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1. Introduction

Due to the diverse range of vibration sources encountered in engineering structures, a variety of different forms of excitation drive vibration. Random inputs constitute a large proportion of realistic excitation, although, depending on the application, the frequency content can range from narrow to broadband. Of particular interest in this paper are structures under stationary Gaussian random excitation of any spectrum.

In a number of applications the power dissipated by a vibrating system will be of interest. In some instances, such as vibration energy harvesting, the aim will be to maximise power, whereas in others, such as those involving concerns over fatigue or heat generation, the aim will be to minimise it. In addition, for single-degree-of-freedom (SDOF) systems with linear damping the power dissipated is proportional to the mean square velocity, a useful measure of the response of an oscillator.

Over the past two decades there has been great interest in vibration energy harvesting [1], the conversion of ambient vibrational energy into electrical energy in order to power small-scale electronic devices. Consequently, a large body of literature has been produced investigating the power dissipation of both linear and nonlinear oscillatory systems under various base

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excitations, although the theory is often applied to specific energy harvesting systems, see for example [2–7].

Of particular interest here are general methods for calculating power dissipation of nonlinear systems from random excitation. In general, the most prevalent technique is to solve the Fokker-Planck equation which governs the probability density function of the response [4,8–10]. Whilst robust, this must be achieved computationally [11] or via simplifying the resulting equations by making assumptions. As the number of degrees of freedom increase, these solutions become significantly more involved. A noteworthy result in the case of power dissipated under white noise excitation is derived partially in Refs. [8,10,12–14], and more generally in Ref. [6] and shows that for a general multi-degree-of-freedom (MDOF) nonlinear system subject to white noise base excitation, the power dissipated is simply proportional to the total oscillating mass and the magnitude of the noise spectrum regardless of the specific details of the system. This result is extended in Ref. [15] which uses the Wiener series to show that for systems exhibiting detailed balance the power dissipated under white noise excitation will be greater than or equal to the power dissipated under non-white excitation where the peak of the spectrum is taken as the magnitude of the white excitation.

The Wiener series is a useful method for analysing an output from a nonlinear system with a Gaussian white noise input via an orthogonal series expansion of the random output [16]. It is a commonly used tool for nonlinear system identification, particularly for physiological systems [17,18]. Whilst generally associated with white noise inputs, it can also be extended to non-white inputs [19] and this form, herein called the *extended* Wiener series, is applied in this paper. A thorough description and explanation of Wiener theory and its applications can be found in Ref. [16].

The aim of this paper is to provide a general methodology for calculating the power dissipated by a general nonlinear oscillator under non-white excitation. In what follows an introduction to the extended Wiener series is presented in Section 2 followed by the derivation of the method for calculating power dissipation in Section 3. The theory is then validated numerically and experimentally in Sections 4 and 5 respectively before conclusions are made in Section 6.

2. Wiener series for non-white excitation

In this section, the extended Wiener series for non-white input excitation is introduced. The series is very similar to the Wiener series for white noise and as such, the notation of [16] is used. A nonlinear system with a Gaussian random input, $x(t)$, will produce a random output signal, $y(t)$, that can be described as a sum of functionals

$$y(t) = \sum_{n=0}^{\infty} \mathbf{g}_n[k_n; x(t)]. \tag{1}$$

When compared to the Wiener series of [16], the G-functionals, $\mathbf{G}_n[k_n; x(t)]$, have been replaced with a lower case $\mathbf{g}_n[k_n; x(t)]$ to represent that these are extended Wiener functionals for non-white noise. Each g-functional is defined as a sum of Volterra functionals, $\mathbf{K}_{j(n)}[x(t)]$, up to the order of the g-functional such that

$$\mathbf{g}_n[k_n; x(t)] = \sum_{j=0}^{n} \mathbf{K}_{j(n)}[x(t)] \tag{2}$$

where the Volterra functional $\mathbf{K}_{j(n)}[x(t)]$ takes the form

$$\mathbf{K}_{j(n)}[x(t)] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} k_{j(n)}(\tau_1, \dots, \tau_j) x(t - \tau_1) \dots x(t - \tau_j) d\tau_1 \dots d\tau_j. \tag{3}$$

The order of the Volterra functional is j and $k_{j(n)}(\tau_1, \dots, \tau_j)$ is called an extended Wiener kernel of order j and must be calculated. The (n) term in the subscript of both the functional and the kernel denotes that they both belong to the Volterra series of the nth order g-functional. When $j = n$ the nth order kernel $k_{n(n)}$ will be rewritten as k_n and is called the leading order kernel.

The relationship between the extended Wiener kernels in a g-functional can be found by enforcing the orthogonality condition

$$\mathrm{E}\left[\mathbf{H}_p[x(t)]\mathbf{g}_n[k_n; x(t)]\right] = 0 \qquad \text{for} \qquad p < n \tag{4}$$

where $\mathrm{E}[X]$ represents taking the ensemble average of the random variable X and $\mathbf{H}_p[x(t)]$ is any Volterra functional of order p, where the form of a Volterra functional is given in Eq. (3). This condition is required to create an applicable orthogonal series, like the Wiener series, that converges and where the contributions from each g-functional can be isolated in order to calculate them.

When the orthogonality condition is applied, the form of the g-functionals is attained and the first three and the nth g-functionals are found as

$$\mathbf{g}_0[k_0; x(t)] = k_0 \tag{5}$$

$$\mathbf{g}_1[k_1; x(t)] = \int_{-\infty}^{\infty} k_1(\tau_1) x(t - \tau_1) d\tau_1 \tag{6}$$

$$\mathbf{g}_2[k_2; x(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_2(\tau_1, \tau_2) x(t - \tau_1) x(t - \tau_2) d\tau_1 d\tau_2 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_2(\tau_1, \tau_2) R_{xx}(\tau_1 - \tau_2) d\tau_1 d\tau_2 \tag{7}$$