Journal of Sound and Vibration 412 (2018) 372–388

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

State-vector formalism and the Legendre polynomial solution for modelling guided waves in anisotropic plates

Mingfang Zheng, Cunfu He*, Yan Lu, Bin Wu

College of Mechanical Engineering and Applied Electronics Technology, Beijing University of Technology, Beijing, 100124, China

ARTICLE INFO

Article history:
Received 14 June 2017
Received in revised form 28 September 2017
Accepted 6 October 2017

Keywords:
Anisotropic plates
Guided waves
State-vector formalism
Polynomial expansion

ABSTRACT

We presented a numerical method to solve phase dispersion curve in general anisotropic plates. This approach involves an exact solution to the problem in the form of the Legendre polynomial of multiple integrals, which we substituted into the state-vector formalism. In order to improve the efficiency of the proposed method, we made a special effort to demonstrate the analytical methodology. Furthermore, we analyzed the algebraic symmetries of the matrices in the state-vector formalism for anisotropic plates. The basic feature of the proposed method was the expansion of field quantities by Legendre polynomials. The Legendre polynomial method avoid to solve the transcendental dispersion equation, which can only be solved numerically. This state-vector formalism combined with Legendre polynomial expansion distinguished the adjacent dispersion mode clearly, even when the modes were very close. We then illustrated the theoretical solutions of the dispersion curves by this method for isotropic and anisotropic plates. Finally, we compared the proposed method with the global matrix method (GMM), which shows excellent agreement.

1. Introduction

Accurate calculation of the dispersion curve is essential for nondestructive evaluation of structures such as auto parts, wind turbine components, and aerospace and aircraft components. Dispersion curves indicate regions of constructive interference, as well as demonstrate the propagation capability of various modes. The information in dispersion curves are of great importance for guided wave testing. The propagation problem of guided wave testing in plate structures is classical. Many researchers made great efforts to solve the dispersion curve in anisotropic plates. Matrix methods [1], the finite element method (FEM) [2], and the semi-analysis finite element method (SAFE) [3] have been developed to solve the dispersion curves of Lamb waves in isotropic and anisotropic plates.

Both Mindlin [4] and Pao [5] adopted potentials to obtain analytical solutions for the isotropic plate. Furthermore, Solie and Auld [6] used the partial wave expansion to draw dispersion curves of the decoupled shear-vertical and longitudinal modes in the cubic material, while Nayfeh and Chimenti [7] investigated the Lamb wave dispersion curves in anisotropic plates with orthorhombic symmetry and monoclinic symmetry. Liu et al. [8] utilized 3D linear elasticity theory and the transfer matrix method to derive the solution of wave propagation in an arbitrary orthotropic laminate. Then, the real parts of the stress and strain were used to analyze the distribution of particle movement in the thickness direction. Meanwhile, Potel

* Corresponding author.
E-mail address: hecunfu@bjut.edu.cn (C. He).

https://doi.org/10.1016/j.jsv.2017.10.010
0022-460X/

and De Belleval [9] introduced the slowness vector to investigate the leaky Lamb wave propagation in monoclinic material using the transfer matrix method, and Verma [10] derived the closed form of symmetric and anti-symmetric Lamb waves in a single monoclinic layer by assuming that the displacements should be a function of thickness.

All of the above studies were based on the transfer or global matrix method (TMM or GMM). Lowe [1] reviewed the matrix method in detail, and a computer program (DISPERSE) was developed to produce dispersion curves using the global matrix method [11]. These approaches use partial waves in the layer to represent displacement and stress fields. When the stress and displacement boundary conditions are applied, a root-finding algorithm extracts the minima in the determinant of the coefficients matrix of the dispersion equations. However, the matrix method is still based on the numerical solutions. Therefore, it is probable that the root-finding algorithm fails to trace a mode (connecting solutions), and it is difficult to avoid the apparent crossing between adjacent branch modes. It should be noted that this can be partially overcome by adjusting the convergence parameters to obtain a better local search. In addition, numerical instability will occur in the dispersion curves when a large frequency-thickness parameter arises; this is the so-called “large *fd*” problem, and is due to the exponential term in dispersion matrices being too large or too small to induce the numerically ill-conditioned matrix.

An interesting approach is FEM, which has also been presented to extract dispersion curves in various structures. One of the main advantages of FEM is that it can study complicated structures. For example, Gavric [12,13] presented a particular finite element program to extract frequency-wavenumbers relation in the I-profile beam free rail. Meanwhile, Manconi [14] obtained the dispersion curves of a laminated foam-cored sandwich by FEM. FEM method was akin to solving the eigenvalue problem of structural vibration, and was implemented on a general FEM software. Furthermore, the vibration modes of various waveguide structures can be analyzed by FEM [15,16], following which dispersion characteristics can be extracted.

FEM is a useful tool to derive dispersion curves, especially for complex materials and structures. FEM uses the piecewise polynomial as a local interpolation function to approximate the solution on small subdomains. Since the piecewise polynomial is a lower order polynomial, FEM generally improves the accuracy of calculations by a dense mesh. When solving the “large *fd*” problem using solid elements, the locking phenomenon will be produced to reduce the solving accuracy of the guided wave problem [17]. However, the Legendre polynomial expansion approximates the solution as a linear combination of continuous functions that are generally nonzero over the domain of the solution. Therefore, the accuracy of the solution is higher than when using FEM.

A different approach has been proposed to solve dispersion characteristics with arbitrary cross-sections: the semi-analytical finite-element method (SAFEM). The SAFEM defines quantities in the frequency-wavenumber domain. SAFEM employs a finite element mesh representation of the cross-section of the waveguide, and treats the wave propagation direction with the harmonic solution [3]. Hence, only a 2-dimensional discretization of the cross-section is required for SAFEM, whereas a 3-dimensional discretization of the entire waveguide is required by FEM; therefore, the computational requirements are significantly lower. Furthermore, the eigenvalue and corresponding eigenvector from the SAFEM solution are used to calculate the wave propagation problem, and the root-searching algorithm is not needed for SAFEM. The SAFEM has been commonly used to obtain dispersion curves of isotropic and composite plates. SAFEM shows its advantage on obtaining propagative solutions when applied to many complex waveguides, such as rods [18], wires [19], and rail road tracks [3,13].

FEM and SAFEM are more stable than matrix methods, and can deal with structures with complex cross-sectional geometry. However, their calculations are quite time consuming, and their accuracy is limited by the precision of the finite element grid.

An alternative that has been recently and successfully used to obtain the dispersion curve is the spectral method [20–22]. The governing equation and boundary conditions can be converted to the wavenumber-frequency domain by transforming time to the frequency domain. Furthermore, the Chebyshev differential matrix approach is proposed to treat spatial operators. This requires importing a set of interpolation points to represent the displacement field, and the solution accuracy and computation complexity are determined by the number of interpolation points. As a result, the solving problem of the dispersion curve is presented as a general eigenvalue problem. This approach has been validated as a useful, stable, and fast method to solve the dispersion curves of planar and tubular structures. Karpfinger et al. [20], Zharnikov et al. [21], and Quintanilla et al. [22] have successfully used this method to handle isotropic and anisotropic materials.

Although many methods have been proposed to investigate dispersion characteristics, more reliable and robust approaches are still needed to enhance the accuracy and efficiency in analyzing the complicated propagation of Lamb waves. The polynomial expansion was proposed to determine the propagation vector of anisotropic plates without solving any transcendental equation. Lefebvre et al. [23] proposed polynomial expansion to solve the dispersion curve. Based on the Legendre polynomial expansion method, the displacement field in layers was expanded by Legendre polynomials, and then substituted into the wave motion equation.

The problem of solving the differential equation can be transformed into the eigenvalue problem. It should be emphasized here that the eigenvalues contain the angular frequency and wave number information. Many researchers used the Legendre polynomial method to investigate wave propagation characteristics for more complicated materials, such as functionally graded materials and magneto-electro-elastic piezoelectric plates and pipes [24–29]. In addition, Sanderson [30] used the N^{th}-order polynomial method to expand the particle displacements of guided waves, and the closed form solutions of the dispersion problem derived for the stiffness and mass matrix were acquired using FEM while the dispersion curve of the guided wave was solved using the matrix eigenvalue problem. Moreover, a power series technique has also been proposed to expand the unknown quantities for solving the dispersion curves of Lamb waves in an functionally graded materials plate