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A reflected wave superposition method for vibration and energy of a travelling string

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ABSTRACT

This paper considers the analytical free time domain response and energy in an axially translating and laterally vibrating string. The domain of the string is either a constant or variable length, dependent upon the general initial conditions. The translating tensioned strings possess either fixed-fixed or fixed-free boundaries. An alternative analytical solution using a reflected wave superposition method is presented for a finite translating string. Firstly, the cycles of vibration for both constant and variable length strings are provided, which for the latter are dependent upon the variable string length. Each cycle is divided into three time intervals according to the magnitude and the direction of the translating string velocity. Applying d'Alembert's method combined with the reflection properties, expressions for the reflected waves at the two boundaries are obtained. Subsequently, superposition of all of the incident and reflected waves provides results for the free vibration of the string over the three time intervals. The variation in the total mechanical energy of the string system is also shown. The accuracy and efficiency of the proposed method are confirmed numerically by comparison to simulations produced using a Newmark-Beta method solution and an existing state space function representation of the string dynamics.

1. Introduction

In the present paper, the lateral vibration of a uniform finite length string modelled with uniform density ρ, which travels over two smooth supports under constant speed v and under a constant tension T is considered using the concept of wave propagation. This model can be used to study the lateral vibration of many manufacturing technologies and devices involving axially travelling materials, such as conveyor belts, elevator cables, power transmission belts and magnetic tapes. Wave propagation and reflection phenomena in one-dimensional wave bearing systems, such as strings, have been studied for many years; it is though still of great research interest due to their theoretical importance and application. For instance, the classical d'Alembert principle was used to study the reflection phenomenon in either an infinite or a semi-infinite stationary string with classical boundary conditions [1,2]. Recently, Akkaya, Gaiko and Van Horssen [3,4] applied the same method to obtain the exact free, linear, lateral vibration of both a stationary [3] and an axially travelling [4] semi-infinite string. Various alternative approaches have been applied to solve and obtain the response of axially moving materials. Yang and Tan [5] studied both a travelling string and beam using a transfer function method, which for the latter considered a damped, axially moving beam over a set of different boundary conditions. Based on the transfer function formulation and wave propagation, Tan and Ying [6] subsequently derived an exact solution for the response of a

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translating string with general boundary conditions. Van Horssen [7] used a Laplace transform method instead, constructing exact solutions of the lateral vibrations in travelling strings due to small lateral vibrations of the supports.

Lee [8] analyzed free vibration of a string with time-varying length, by dealing with travelling waves and obtained an exact solution. Simple models which describe these vibrations can be expressed as initial-boundary value problems. Darmawijoyo [9,10] studied such an initial-boundary value problem with a non-classical boundary condition, constructing asymptotic approximations of the solution for an axially travelling string by a multiple-timescales perturbation method. Chen and Ferguson [11] more recently studied the lateral vibration and the energy dissipation of a travelling string attached to a viscous damper at one end, using a time varying state space function method and the Newmark-Beta method. In terms of dissipative boundaries, Gaiko and Van Horssen [12] also gave a complete and accurate description of the damping and the low frequency oscillatory behaviour of the travelling string with an attached spring–mass–dashpot system at one end.

In contrast to previous work on a stationary string and a travelling string defined on an infinite or a semi-finite domain, which either has one or no wave reflections, the present work focuses on the analytical free lateral vibration of an axially translating string with either a constant or varying length. The main difference, as well as the difficulty, lies in the multiple reflections that will exist in a finite domain, which makes the problem more complicated than previous work. A reflected wave superposition method is proposed and completely developed in this study. At both ends of the axially travelling string the multiple reflections of the propagating waves are studied. The combined total superposition of the incident and reflected waves constitute the resulting free vibration. This work provides an analytical methodology to solve the translating media problem defined over a finite domain with different boundary conditions and the details of the process are given. This paper is organized as follows. Section 2 introduces the equation of motion describing the lateral vibration of an axially travelling finite string and the relevant boundary conditions. In Section 3, the reflected wave superposition method is used to obtain the response due to the initial conditions of a travelling string between two types of boundaries separated by a constant distance, i.e. the length of string between the boundaries is also constant. Next, Section 4 investigates the time varying cycle and gives an exact analytical response for the time varying length string, i.e. for boundaries that are at either increasing or decreasing distances apart. Section 5 analyzes the total mechanical energy and its time rate of change. Finally, Section 6 provides the main conclusions.

2. Equations of vibration

The equation of motion for the lateral vibration of a travelling string between two boundaries can be obtained by Hamilton Principle [12,13] and is given by

$$\frac{\partial^2 w}{\partial t^2} + 2v\frac{\partial^2 w}{\partial t \partial x} + \left(v^2 - c^2\right)\frac{\partial^2 w}{\partial x^2} = 0 \tag{1}$$

where w is the lateral displacement, x is the axial coordinate along the length of the string, t is the time, v is the assumed constant translational speed of the string. $c = \sqrt{T/\rho}$ is the free wave propagation speed, where T is the uniform tension and ρ is the uniform string mass per unit length. The derivation of Eq. (1) is given in Appendix A. The domain for the string x is $0 < x < l(t)$ and $l(t)$ is the length of the string. Here, two cases concerning the string length are considered: one is the constant length case, i.e. $l(t)=l_0$ and so $v \neq \dot{l}$; the other is the linearly changing length case, i.e. $l(t)=l_0+vt$, and so $v = \dot{l}$, where l_0 is the initial length of the string. To avoid the vibrational energy accumulating at one end and to allow any propagating wave in the string direction to be reflected at the other end, the string translational speed v is assumed less than the free wave propagation speed c, i.e. $|v| < c$.

Various boundary conditions can exist in real situations, for example, free, fixed, spring-dashpot and mass-spring-dashpot, etc. Although only classical boundary conditions are analyzed in this paper, i.e. the free end and the fixed end, the proposed method can also be applied to the nonclassical boundary conditions.

The general one-dimensional wave solution of Eq. (1) for the string displacement using the d'Alembert method is well known [2,14]. It is given by

$$w(x, t) = F(x - v_r t) + G(x + v_l t) \tag{2}$$

Here, $v_l=c-v$ is the wave propagation speed according to the fixed coordinate system as a wave travels from right to left and $v_r=c+v$ is the wave speed in the opposite direction. $F(x-v_r\, t)$ is the right-propagating wave with speed of v_r and.

$G(x+v_l t)$ is the left-propagating wave with speed of v_l. In Appendix B, it is shown that the d'Alembert method can also be used as a solution for the system with a governing equation of motion as Eq. (1). The initial vibration conditions for the string displacement and velocity are assumed as follows:

$$\begin{cases} w(x, 0) = \varphi(x) \\ w_t(x, 0) = \psi(x) \end{cases}, \qquad (0 \le x \le l(t)) \tag{3}$$

where w_t is the first order partial derivation of $w(x, t)$ with respect to time t. Then, the initial conditions are satisfied if one has

$$F(x) = \frac{1}{2c}v_l\varphi(x) + \frac{1}{2c}\int_x^{l(t)} \psi(\xi)d\xi + K \tag{4}$$