Journal of Sound and Vibration 400 (2017) 248–269

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

The one-dimensional acoustic field in a duct with arbitrary mean axial temperature gradient and mean flow

Jingxuan Li, Aimee S. Morgans *

Department of Mechanical Engineering, Imperial College London, London, UK

ARTICLE INFO

Article history:
Received 26 October 2016
Received in revised form
30 March 2017
Accepted 31 March 2017
Handling Editor: R.E. Musafir

Keywords:
Duct acoustics
WKB approximation
Entropy waves
Piecewise linear approximation

ABSTRACT

This paper presents an analytical solution for the one-dimensional acoustic field in a duct with arbitrary mean temperature gradient and mean flow. A wave equation for the pressure perturbation is derived which relies on very few assumptions. An analytical solution for this is derived using an adapted WKB approximation. The solution is a superposition of waves travelling in either direction and thus provides physical insight. It is also very easy to calculate. The proposed solution is applied to ducts with a mean temperature profile which varies axially with (i) a linear and (ii) a partial sine wave profile: predictions are compared to those obtained by numerically solving the linearised Euler equations (LEEs). The analytical solution reproduces the acoustic field very accurately across a wide range of flow conditions which span both low and moderate-to-high subsonic Mach numbers. It always performs well when the frequency exceeds a certain value; when the mean temperature profile is linear, it also performs well to very low frequencies. This increased frequency range for linear mean temperature profiles leads to its application to more complicated profiles in a piecewise linear manner, axially segmenting the temperature profile into regions that can be approximated as linear. The acoustic field is predicted very accurately as long as enough segmentation points are used and the condition for the linear mean temperature profile is satisfied: $|k_0| > |\alpha|$, where k_0 is the local wave number when there is no mean flow and α is the normalised mean density gradient. The proposed solution is extensively compared to previous analytical solutions, and is found to be more accurate and reliable, especially at higher Mach numbers. The entropy wave generated by communication between the acoustic waves and the distributed mean temperature zone is calculated using the LEEs. It is found to remain very small across all operating conditions, such that both the entropy wave and its impact on the acoustic field can be neglected.

1. Introduction

Ducts sustaining both a mean flow and a mean axial temperature gradient are a common element of engineering systems, including gas turbine combustion chambers [1–4] and exhaust systems [5]. An accurate analytical solution for the one-dimensional acoustic field within such ducts would be valuable, particularly if it could be represented as the

* Corresponding author.
E-mail addresses: jingxuan.li@imperial.ac.uk, ljx.buaa@gmail.com (J. Li), a.morgans@imperial.ac.uk (A.S. Morgans).

http://dx.doi.org/10.1016/j.jsv.2017.03.047

superposition of waves travelling in either direction. This would offer enhanced physical insight, reduce the computational cost of numerical tools used to predict [6–10] and control [11,12] thermoacoustic instabilities, and would bring benefits for measurement approaches based on the two-microphone [13,14], and reproduction of underwater thermodynamic properties based on the solution of the inverse scattering problem in one dimension [15,16].

Approaches for deriving the analytical solution of the one-dimensional acoustic field generally fall within two categories. The first is based on variable transformation. The wave equation is transformed to a standard second order ordinary differential equations with known solutions. For example, Sujith and co-workers transformed the acoustic wave equation to Bessel functions for linear and exponential mean temperature profiles [17]. This method was extended to derive exact analytical solutions for quadratic [18] and polynomial mean temperature profiles [19] by the same research group. However, all above solutions are limited to zero and very low Mach number mean flows. Davies [20] derived a wave equation as a function of velocity perturbation based on the assumption of small linear mean temperature gradient and low Mach number mean flow. The wave equation was transformed to a confluent hypergeometric function, but was eventually solved numerically for most operating conditions due to convergence problems with the confluent hypergeometric series. Karthik et al. [21] derived an acoustic wave equation for low Mach number mean flow. By further assuming a linear mean temperature profile, they transformed the acoustic wave equation to a Gauss hypergeometric equation whose fundamental solutions are two Gauss hypergeometric series. However, this method is limited to low Mach number flow and linear mean temperature profiles. Furthermore, convergence problems occur for large frequencies and small mean temperature gradients.

The second category uses linear perturbation theory and assumes that the acoustic wave equation consists of wave-like solutions for slowly varying coefficients of the ordinary differential equation [22]. The solution can generally be expressed as an exponential. Munjal and Prasad [23] derived a wave equation as a function of pressure perturbation, assuming small linear mean temperature gradients and low Mach number mean flow. An analytical solution was obtained based on linear perturbation theory and the Green's function approach. However, the gradients of mean density and velocity along the duct were wrongly neglected in the wave equation, leading to inaccuracies. Peat [5] improved the wave equation to retain these terms and assumed an exponential solution, which was the superposition of a base solution for no mean temperature gradient and a small linearised perturbation solution. By balancing terms of different orders, he obtained an analytical solution. Similar approximate analytical solutions were obtained by Dokumaci [24] based on linear perturbation theory and a matrizant formulation of pressure and velocity perturbations. However, the above solutions are all limited to small linear mean temperature gradients and low mean flow Mach numbers. Cummmings [25] derived an analytical solution using an adapted WKB approximation and assuming sufficiently large frequencies and the absence of mean flow. Subrahmanyam et al. [26] used a similar method to derive a family of exact time-domain travelling wave-type solutions in ducts with mean temperature and area variations in the absence of mean flow. This WKB approximation method was extended to account for mean flow [27]. However, too many terms were omitted in the wave equation and solutions are not accurate at larger mean flow Mach numbers. To summarise, no analytical or semi-analytical solution for the one-dimensional acoustic field in a duct has previously been presented which allows an arbitrary mean axial temperature gradient and a mean flow of moderate subsonic Mach number.

This work derives an acoustic wave equation which relies on very few assumptions. It then uses an adapted WKB approach to derive the analytical solution for the one-dimensional acoustic field. The proposed analytical solution is simple and applies to large mean temperature gradients and moderate-to-large mean flow Mach numbers. Furthermore, for linear mean temperature profiles, it is seen to yield a particularly simple expression and to be accurate at both low and high frequencies. This suggests that accurate prediction for arbitrary mean temperature profiles can be achieved by applying it in a piecewise linear manner to an appropriately axially segmented mean temperature profile.

The remainder of the paper is organised as follows. The derivations of the acoustic wave equation and analytical solutions are presented in Section 2 and 3 respectively. Section 4 introduces the linear and sine wave mean temperature profiles and the two transfer functions used for validation of the proposed analytical solution. Validation of predictions for the linear mean temperature profile are presented in Section 5, and for the sine wave mean temperature profile in Section 6. The entropy wave generated by the communication between the acoustic waves and the distributed mean temperature zone, and its effect on the acoustic field are quantitatively investigated in Section 7. The main equations to calculate the mean thermodynamic properties within the duct are presented in Appendix A. Because the solution for the velocity perturbation was not provided in Cummings' paper [27], it is derived in this work and is presented in Appendix B. Karthik's solutions have been corrected and simplified for use – these are presented in Appendix C. Conclusions are drawn in the final section.

2. Acoustic wave equation

In general, previous work directly derives the acoustic wave equation from the mass and momentum equations, assuming that there is no thermal conductivity of the fluid (the isentropic assumption) [27,5,24,21]. The entropy is thereby directly neglected. In this section, the acoustic wave equation is derived step by step. A constant cross-sectional area duct sustaining a mean flow and mean temperature gradient is considered. Assuming a perfect inviscid gas, the one-dimensional mass, momentum and energy conservation equations along with the perfect gas law along the duct give [28]: