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On a comparative study of an accurate spatial discretization method for one-dimensional continuous systems

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ABSTRACT

This work provides an in-depth investigation on advantages of a recently developed, new global spatial discretization method over the assumed modes method, and a clear description of the procedure and validity of the new method and its feasibility for arbitrary boundary conditions. A general formulation of the new spatial discretization method is given for second- and fourth-order continuous systems, whose displacements are divided into internal terms and boundary-induced terms, and two examples that consider the longitudinal vibration of a rod and the transverse vibration of a tensioned Euler-Bernoulli beam are used to demonstrate the new spatial discretization method. In the two examples, natural frequencies, mode shapes, harmonic steady-state responses, and transient responses of the systems are calculated using the new spatial discretization method and the assumed modes method, and results are compared with those from exact analyses. Convergence of the new spatial discretization method is investigated using different sets of trial functions for internal and boundary-induced terms. While the new spatial discretization method has additional degrees of freedom at boundaries of a continuous system compared with other global spatial discretization methods, it has the following advantages: (1) compared with the assumed modes method, the new method gives better results in calculating eigensolutions and dynamic responses of the system, and allows more terms to be retained in a trial function expansion due to the slowly growing condition number of the mass matrix of the system; and (2) compared with the exact eigenfunction expansion method, the new method can use sinusoidal functions as trial functions for the internal term rather than complicated eigenfunctions of the system in the expansion solution.

1. Introduction

A vibrating system is usually studied using either a discrete or continuous system model [1]. A discrete or lumped-parameter system model has a finite number of degrees of freedom and a set of ordinary differential equations (ODEs) as its governing equations. A continuous or distributed-parameter system model has an infinite number of degrees of freedom and one or more partial differential equations (PDEs) as its governing equations, whose independent variables include spatial and temporal variables. Dynamic response of a discrete system is directly calculated using an ODE solver, and only

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Nomenclature

α, β, γ	prescribed coefficients of a partial differential equation
x	independent spatial variable
t	independent temporal variable
u	dependent variable
x_1, x_2	boundary locations
λ	eigenvalue
λ_j	j-th eigenvalue
Φ_j	j-th eigenfunction
ϕ_j	j-th trial function
q_j	j-th generalized coordinate
$\tilde{u}$	internal term
$\hat{u}$	boundary-induced term
φ_j	j-th trial function for the internal term
p_j	j-th generalized coordinate for the internal term
θ_i	i-th interpolation function for boundary degrees of freedom
e_i	i-th boundary degree of freedom
N	number of truncated terms
K	number of additional boundary degrees of freedom
l	length of a continuous system
ρ	linear density of a continuous system
EA	axial stiffness of a second-order system
m	mass of a lumped sub-system
k	stiffness of a lumped sub-system
F	external force
$\mathbf{M}^r$	mass matrix of a second-order system
$\mathbf{K}^r$	stiffness matrix of a second-order system
$\boldsymbol{\phi}$	vector of trial functions
$\mathbf{F}^r$	force vector of a second-order system
ω	natural frequency
ω_j	j-th natural frequency
$\boldsymbol{\Phi}$	vector of eigenfunctions
F_0	amplitude of the external harmonic force
M_0	amplitude of the external harmonic moment
ω_F	frequency of the external harmonic force and moment
A_0	amplitude of the steady-state displacement at $x=l$
$\mathbf{q}$	vector of generalized coordinates
P	tension in a continuous system
EI	bending stiffness of a fourth-order system
I_m	rotatory inertia of a lumped sub-system
M	external moment
$\mathbf{M}^{tb}$	mass matrix of a fourth-order system
$\mathbf{K}^{tb}$	stiffness matrix of a fourth-order system
$\mathbf{F}^{tb}$	force vector of a fourth-order system
I, J	numbers of spatial and temporal nodes in the finite difference method
ξ, η	spatial and temporal steps in the finite difference method
c	damping coefficient

displacements of the system and their time derivatives are obtained. Proper spatial discretization methods are needed for a continuous system so that its governing PDEs can be converted to a set of ODEs and dealt with using an ODE solver. Not only displacements of the system, but also their spatial derivatives, are of interest, since they are directly related to internal forces and moments of the system [2,3]. There are in general two classes of spatial discretization methods: local and global methods. The local methods, such as finite difference [4] and finite element [5] methods, discretize a spatial domain into multiple smaller domains, and evaluate governing equations at connecting points between them. The finite element method can be combined with the transfer-matrix method to deal with wave propagation in a periodic structure [6]. The global methods, such as Trefftz [7], assumed modes, and Galerkin's methods, discretize a spatial domain using a set of trial functions. Trefftz method approximates a solution by a superposition of trial functions that satisfy a governing equation, with its unknown coefficients determined to satisfy boundary conditions [7]. The wave-based method is an indirect Trefftz method that can be used to calculate natural frequencies and mode shapes of a system [8]. The assumed modes method uses admissible functions that satisfy only geometric boundary conditions of a system as its trial functions, and Galerkin's method uses comparison functions that satisfy both geometric and natural boundary conditions as its trial functions [1,9]. Exact analysis can be performed on a system using the eigenfunction expansion method with trial functions in Galerkin's method being eigenfunctions of the system. However, Galerkin's method and exact analysis are not always feasible for systems with complicated boundary conditions, since comparison functions and eigenfunctions that satisfy these boundary conditions cannot be easily determined. Unlike assumed modes and Galerkin's methods that can be used to determine dynamic response of a continuous system, Rayleigh-Ritz method is based on stationarity of Rayleigh's quotient and deals with the eigenvalue problem of the system; it cannot be used to determine its dynamic response.

A new global spatial discretization method that can ensure all internal and boundary conditions of one-dimensional continuous systems are satisfied is recently developed [2] and used to calculate displacements and their spatial derivatives of moving elevator cable-car systems, the latter of which are related to tensions, bending moments, and shear forces of the cables [3]. The new spatial discretization method discretizes a continuous system with complicated boundary conditions by separating a displacement of the system into an internal term and a boundary-induced term [2,3], where the internal term satisfies certain prescribed simple homogeneous boundary conditions and the boundary-induced term accounts for corresponding boundary conditions that are not satisfied by the internal term using additional degrees of freedom at boundaries of the system, whose number does not exceed the number of boundary conditions of the system. The methodology is applicable to both linear and nonlinear systems and uniform convergence of solutions is shown in Ref. [2]. The new

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