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A modal-based approach to the nonlinear vibration of strings against a unilateral obstacle: Simulations and experiments in the pointwise case

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ABSTRACT

This article is concerned with the vibration of a stiff linear string in the presence of a rigid obstacle. A numerical method for unilateral and arbitrary-shaped obstacles is developed, based on a modal approach in order to take into account the frequency dependence of losses in strings. The contact force of the barrier interaction is treated using a penalty approach, while a conservative scheme is derived for time integration, in order to ensure long-term numerical stability. In this way, the linear behaviour of the string when not in contact with the barrier can be controlled *via* a mode by mode fitting, so that the model is particularly well suited for comparisons with experiments. An experimental configuration is used with a point obstacle either centered or near an extremity of the string. In this latter case, such a pointwise obstruction approximates the end condition found in the tanpura, an Indian stringed instrument. The second polarisation of the string is also analysed and included in the model. Numerical results are compared against experiments, showing good accuracy over a long time scale.

1. Introduction

The problem of vibrating media constrained by a unilateral obstacle is a longstanding problem which has been under study for more than a century [1,2]. Indeed, the first important developments can be attributed to Hertz with his formulation of a general law for the contact between elastic solids in 1881 [3]. Since then, applications of contact mechanics can be found in such diverse fields as *e.g.* computer graphics [4], for instance for simulating the motion of hair [5]; to human joints in biomechanics [6] or component interactions in turbines [7,8]. A particular set of applications is found in musical acoustics, where collisions are of prime importance in order to fully understand and analyse the timbre of musical instruments [9–11]. Within this framework, the problem of a vibrating string with a unilateral constraint, as a key feature of numerous instruments, is central and is particularly important to the sound of Indian instruments [12–14], and also in the string/fret contact in fretted instruments [15,16].

The first studies on a vibrating string with a unilateral constraint were restricted to the case of an ideal string with a rigid obstacle in order to derive analytical and existence results [17–21]. In particular, solutions to the cases of a centered point obstacle,

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a plane obstacle and a few continuous obstacles have been obtained explicitly. Existence and uniqueness of the solution to the non-regularised problem have been shown in the case of a string vibrating against point obstacles [22] and of a concave obstacle if conservation of energy is imposed [20]. There are no general results when the obstacle is convex. Moreover, Schatzman proved that the penalised problem with a point obstacle converges to the non-regularised problem [22]. The pointwise case is thus well-understood theoretically, and various interesting properties have been demonstrated as mentioned above.

In addition, numerical studies have been undertaken to simulate collisions for more realistic string models by including various effects, such as dispersion. Existing numerical methods include digital waveguides [23–25], sometimes coupled with finite differences [26] in the case of an ideal string, and for a stiff damped string interacting with an obstacle located at one end of the string [14]. Other models are based on a modal description of the string motion, as in [27] where an ideal string vibrating against a parabolic obstacle at one boundary is considered, under the assumption of perfect wrapping of the string on the bridge as in [28]. However, the existence of multiple contacts as a necessary condition for simulating the sound of sitar has been established in [27,29]. Contacts between a string and point obstacles are modelled with a modal approach in [30] for a dispersive lossy string against a tanpura-like bridge. The functional transformation method (FTM) is used in [15] for a string interacting with frets. In the latter study, the damping model is controlled by a few parameters only. Interaction between a continuous system and a point obstacle is also modelled in [29], using a modal coefficient of restitution (CoR) method [31,32], assuming infinitesimal contact times.

More recently, energy-based methods have been developed, allowing the simulation of stiff lossy strings against an arbitrarily shaped obstacle. Hamilton's equations of motion are discretised in [33], and the case of the tanpura bridge is derived in [34]. Finite difference methods are used in [11] and the special case of the interaction between a string and a fretboard is detailed in [16]. In these latter models, eigenfrequencies and damping parameters cannot be arbitrary, but follow a distribution tuned through a small number of parameters. In addition, these studies consider only one transverse motion of the string, and numerical dispersion effects appear due to the use of finite difference approximations.

The inclusion of the two transverse polarisations in the modeling of vibrating strings with contact is seldom seen in the literature. A first attempt has been proposed in [35] for the case of the violin, where finite differences are employed to model a linear bowed string motion, including interactions between the string and fingers as well as the fingerboard. Early developments are also shown in [36], extending the study presented in [33]. However, numerical results are not compared to experimental measurements of the string motion.

Whereas an abundant literature exists on numerical simulations of a string vibrating against an obstacle, only a few experimental studies have been carried out. Research on isolated strings is detailed in [37,30], and measurements on complete instruments are presented in [38,39,14], highlighting the influence of the obstacle shape and position on the timbral richness of sounds. However a detailed comparison of experiments with numerical results in order to understand the relative importance of modeling features such as *e.g.* dispersion, nonlinearity and damping due to contact has not been carried out.

The aim of this paper is twofold. First, an accurate and flexible numerical method is developed in Section 2. The distinctive feature of the approach is that it relies on a modal description, in order to take into account any frequency dependence of the losses, and also in order to eliminate any effect of numerical dispersion. The contact law is formulated in terms of a penalty potential and an energy-conserving scheme is derived, adapted to the modal-based approach. The convergence of the outcomes of the models is then thoroughly studied in Section 3 for a pointwise obstacle, with a comparison to an analytical solution. The second main objective of the study is to compare simulations with experiments. For that purpose, the experimental protocol is presented in Section 4. The versatility of the numerical method is illustrated with a mode by mode fitting of the measured linear characteristics (eigenfrequencies and modal damping factors). Comparisons with experiments are conducted in Section 5 for two different point obstacles, located either at the string centre or near one extremity of the string. The second polarisation is also measured and compared to the outcomes of a simple model incorporating the horizontal vibration in Section 5.2.3.

2. Theoretical model and numerical implementation

2.1. Continuous model system

The vibrating structure considered here is a stiff string of length L (m), tension T (N.m^{-1}), and with linear mass density μ (kg.m^{-1}). The stiffness is described by the Young's modulus E (Pa) of the material and the moment of inertia associated with a circular cross-section $I = \pi r^4/4$, where r is the string radius (m). The string is assumed to vibrate in the presence of an obstacle described by a fixed profile $g(x)$, $x \in [0, L]$, located under the string at rest (see Fig. 1). The obstacle is assumed to be of constant height along (Oy). In the remainder of the paper, it is said to be a *point obstacle* when it is a point along (Ox), however it still has a constant height along (Oy).

In this section we restrict ourselves to the vertical (Oz)-polarisation. The second, horizontal polarisation is taken into account in Section 2.7.

The transverse displacement $u(x, t)$ of the string along (Oz) is governed by the following equation, under the assumption of small displacements:

$$\mu u_{tt} - T u_{xx} + EI u_{xxxx} = f, \tag{1}$$

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