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Dynamic equations for an isotropic spherical shell using the power series method and surface differential operators

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ABSTRACT

Dynamic equations for an isotropic spherical shell are derived by using a series expansion technique. The displacement field is split into a scalar (radial) part and a vector (tangential) part. Surface differential operators are introduced to decrease the length of all equations. The starting point is a power series expansion of the displacement components in the thickness coordinate relative to the mid-surface of the shell. By using the expansions of the displacement components, the three-dimensional elastodynamic equations yield a set of recursion relations among the expansion functions that can be used to eliminate all but the four of lowest order and to express higher order expansion functions in terms of those of lowest orders. Applying the boundary conditions on the surfaces of the spherical shell and eliminating all but the four lowest order expansion functions give the shell equations as a power series in the shell thickness. After lengthy manipulations, the final four shell equations are obtained in a relatively compact form which are given to second order in shell thickness explicitly. The eigenfrequencies are compared to exact three-dimensional theory with excellent agreement and to membrane theory.

1. Introduction

Shells are commonly used in many branches of engineering with applications such as pressure vessels, fuselages of airplanes, boat and ship hulls, roof structures, bodies of cars, trains, and aeroplanes. A shell can be considered as a curved plate having a small thickness compared to the other geometrical dimensions as well as to the wavelengths of importance. The most important advantage of shells in comparison to plates is that shell structures can provide high strength and low weight because of their membrane stiffness. Governing equations for spherical and cylindrical shells are given by Soedel [1], and thin cylindrical shells are studied by Leissa [2]. Basic equations for spherical and cylindrical shells are discussed by Niordson [3].

Spherical shells appear in some structures and dynamic shell theories have thus been developed for this case. Most of these theories seem to depend on more or less ad hoc kinematical assumptions and/or other approximations. For the present purposes the most relevant references seem to be those of Shah et al. [4,5] and Niordson [6]. Shah et al. [4] seem to be the first to give the exact three-dimensional solution for the eigenfrequencies of a spherical shell (of arbitrary thickness), drawing on earlier work by Morse and Feshbach [7]. They also give a higher-order bending theory including a shear correction factor and relevant references to the older literature. Niordson [6] uses an asymptotic method that has similarities (but also differences) to the present approach. This method also has the benefits of not using any ad hoc assumptions and

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the possibility to go to, in principle, any order in the thickness. A more recent investigation into the vibrations of spherical structural elements is the work by Tornabene and Viola [8], which use a first order shear deformation theory (similar to the one used by Shah et al. [4]) and a numerical technique known as the GDQ method to compute eigenfrequencies and mode shapes for spherical panels.

In this paper dynamic equations for a spherical shell are derived by using a power series method developed during the last decade, first for a rod by Boström [9]. (Martin [10,11] use a similar method to investigate cylindrical anisotropic rods.) The method has also been developed for plates, see Boström et al. [12] and Mauritsson et al. [13,14] and to an isotropic cylindrical shell, see Hägglund and Folkow [15]. The method is very systematic and very useful for developing the equations to any order. By increasing the order, results with a very high accuracy can be obtained. It should be stressed that the main goal of the present paper is the derivation of the (hierarchy of) shell equations and to validate these against exact 3D solutions for a full spherical shell. The edge boundary conditions needed for a spherical panel or the like are thus not taken up, as is not the numerical solution of the equations either (by a FEM discretization or otherwise).

The dynamic equations for an isotropic spherical shell have previously been derived by the power series approach by Okhovat and Boström [16], but all derivatives and components were explicitly written out so the equations could only be written to lowest order (membrane equations), due to the forbiddingly lengthy expressions that result to the next order. Here this difficulty is tackled by introducing surface differential operators and combining the tangential field components into a single (two-component) vector field.

The present paper can be described as follows. In the next section the problem is stated and the three-dimensional equations of elasticity are given. The displacement field is split into scalar (normal) and vector (tangential) parts and surface differential operators are introduced. Next, the expansion of the displacement components in series in the thickness coordinate is performed, leading to the recursion relations for the expansion functions. By using the recursion relations, all the higher order expansion functions can be expressed in terms of some of the lowest ones. This is the key ingredient in the present approach. Applying the boundary conditions at the inner and outer surfaces of the spherical shell and using the recursion relations to eliminate all but the four lowest-order expansion functions give the four shell equations. These can in principle be given to any order and it is noted that the boundary conditions are satisfied exactly for any order. The lengthy analytical calculations are performed using the commercial program Mathematica.

Numerical results for eigenfrequencies for a full spherical shell are computed by using the present approach for different shell thicknesses and for different orders of approximation. The results are compared with the results given by the exact solution. The exact solution is obtained by using the spherical partial vector wave solutions to the elastodynamic equations of motion and by expressing the displacements in terms of spherical Bessel and Neumann functions. The procedure is described in the Appendix. Shah et al. [4] also give the exact solution, but by writing everything in components which lead to a more cumbersome and less transparent approach. Computations of the eigenfrequencies from the shell equations are in excellent correspondence with the results from the exact solution.

2. Problem formulation

Consider a spherical shell with mean radius R and thickness $2\,h$. The material is assumed to be isotropic and linearly elastic with Lamé constants λ and μ and density ρ. Introduce spherical coordinates r, θ, and φ, where r is the radial coordinate, θ the polar coordinate, and φ the azimuthal coordinate. The main goal is to derive a set of dynamic shell equations for this case, i.e. a set of differential equations that depend on the two angular spherical coordinates and time, but where the radial dependence has disappeared.

The starting point is the three-dimensional dynamic equations of elasticity for the displacement field $\mathbf{u}$ (Achenbach [17])

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}=\rho\partial_t^2\mathbf{u}, \tag{1}$$

where ∂_t is the partial derivative with respect to time. Likewise, ∂_r, ∂_θ, and ∂_φ denote derivatives with respect to the space coordinates. In spherical coordinates this equation is written for the radial component w, the polar component u, and the azimuthal component v.

To decrease the lengths of all equations and to make the approach more transparent both the fields and the derivatives are split into radial and tangential components. This will, in particular, reduce the length of the final shell equations drastically. First the displacement field is split into a scalar (radial) part and a vector (tangential) part as follows

$$\mathbf{u}=w\,\boldsymbol{e}_r+\mathbf{S}, \tag{2}$$

$$\mathbf{S}=u\,\boldsymbol{e}_\theta+v\,\boldsymbol{e}_\varphi, \tag{3}$$

where $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_\varphi$ are the unit vectors in the r, θ, and φ directions, respectively. The various differential operators are treated in a similar fashion. Considering the form of the gradient, the divergence, and the vector Laplacian in spherical coordinates it is natural to define the following surface differential operators (on the unit sphere)

$$\nabla_a w=\mathbf{e}_{\boldsymbol{\theta}}\partial_\theta w+\frac{\mathbf{e}_\varphi}{\sin\theta}\partial_\varphi w, \tag{4}$$

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