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Continuation of quasi-periodic solutions with two-frequency Harmonic Balance Method

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ABSTRACT

The continuation of quasi-periodic solutions for autonomous or forced nonlinear systems is presented in this paper. The association of the Asymptotic Numerical Method, a robust continuation method, and a two-frequency Harmonic Balance Method, is performed thanks to a quadratic formalism. There is no need for *a priori* knowledge of the solution: the two pulsations can be unknown and can vary along the solution branch, and the double Fourier series are computed without needing a harmonic selection. A norm criterion on Fourier coefficients can confirm *a posteriori* the accuracy of the solution branch. On a forced system, frequency-locking regions are approximated, without blocking the continuation process. The continuation of these periodic solutions can be done independently. On an autonomous system an example of solution is shown where the number of Fourier coefficients is increased to improve the accuracy of the solution.

1. Introduction

Time-periodic solutions are commonly investigated in dynamical systems. They can be a source of nuisance, causing noise or potentially destructive motions (for example in rotating machinery) or they can be sought after (for example in many musical instruments relying on auto-oscillations: winds, bowed strings, etc). The well-known continuation of periodic solutions aims at producing bifurcation diagrams representing the existence, stability and other characteristics of such periodic solutions (amplitude, pulsation, etc) with respect to some parameters of the nonlinear system.

However, other solutions can arise, among which, quasi-periodic solutions. Our aim in this paper is the continuation of two-pulsation, quasi-periodic solutions. The direct computation of quasi-periodic solutions through numerical integration can be difficult. Because of dependence on initial conditions, some solutions may be overlooked (as in the periodic case). Moreover, performing integration on long intervals to get rid of transient solutions - a brute force, expensive tactic when studying periodic solutions - needs a new kind of stopping criterion since one cannot easily determine if the steady-state solution is reached when non periodic solutions are considered. These drawbacks have lead to different approaches to the study of quasi-periodicity.

Chua and Ushida [1] adapted the Harmonic Balance Method (HBM) to compute the steady-state response to a quasi-periodic forcing. Another early work based on a frequency-domain approach was the incremental HBM devised by Lau et al.

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[2]. Kaas-Petersen [3,4] reformulated the problem in the time domain in terms of Poincaré maps. An interesting advantage to this approach is that it provides a stability analysis [5,6].

The Alternating Frequency-Time method proposed by Cameron [7] was an important breakthrough, allowing the consideration of strong nonlinearities by computing them in the time domain. Several works used it to find quasi-periodic solutions [8] and coupled it with arc-length continuation [9,10].

Finally, instead of computing a trajectory on an invariant tori, the continuation of these tori was performed independently by Rasmussen [11] and Schilder et al. [12,13].

The method of obtaining and continuing quasi-periodic solutions presented in this paper is different and original. The aim is to create a resilient method that is capable of coping with either forced or autonomous systems, and is fast enough to tackle algebraic systems with a few thousand unknowns. The association of the Asympotic Numerical Method (ANM), a continuation technique based on Taylor series, with the Harmonic Balance Method (HBM), proved its effectiveness to continue periodic solutions of forced or autonomous systems [14]. This framework requires a quadratic formalism, which can be obtained for most problems thanks to auxiliary variables. The present paper extends the principle of coupling ANM and HBM to the quasi-periodic case. The ANM is briefly recalled in Section 2. It is a very robust continuation method and its choice, differing from the widespread arc-length continuation, aims to deal appropriately with difficult situations, for instance frequency-locking.

Several aspects of the HBM are presented in Section 3. After a general sketch of its association with the ANM, an original implementation of the periodic HBM is given in Section 3.2. It is based on the complex representation of the Fourier basis. Then, its adaptation for quasi-periodic solutions is presented in Section 3.3. The method is capable of dealing with periodically forced systems as well as autonomous systems, illustrated respectively in sections 4 and sections 5. In this last case, the two pulsations are both unknown. The method presented here does not need to select specific spectral contributions, and to the authors' knowledge it is able to compute solutions with more harmonics than existing methods.

2. Asymptotic numerical method (ANM)

Let us consider a vector-valued, smooth function:

$$\mathbf{R}: \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^N, \quad (\mathbf{X}, \lambda) \mapsto \mathbf{R}(\mathbf{X}, \lambda) \tag{1}$$

Assuming that a first regular solution $(\mathbf{X}_0, \lambda_0)$ is known, the aim of the Asymptotic Numerical Method (ANM) is to follow the solution branch of the equation:

$$\mathbf{R}(\mathbf{X}, \lambda) = 0 \tag{2}$$

with respect to λ. Without a great loss of generality $\mathbf{R}$ can be considered quadratic, meaning that there are $\mathbf{C}_0, \mathbf{C}_1 \in \mathbb{R}^N$, $\mathbf{L}_0, \mathbf{L}_1$ real $N \times N$ matrices, $\mathbf{Q}_0$ a bilinear application of $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, so that:

$$\mathbf{R}(\mathbf{X}, \lambda) = \mathbf{C}_0 + \lambda \mathbf{C}_1 + \mathbf{L}_0 \mathbf{X} + \lambda \mathbf{L}_1 \mathbf{X} + \mathbf{Q}_0(\mathbf{X}, \mathbf{X}) \tag{3}$$

As noted when the method was designed, the transformation from a general function $\mathbf{R}$ to the quadratic formalism required in Eq. (3) is possible for usual smooth functions, adding auxiliary variables [15–17]. The continuation parameter is now treated as an unknown; thus, a vector of all unknowns is defined as $\mathbf{V} := (\mathbf{X}, \lambda)$, and the idea of the ANM is to develop the solution branch as a (truncated) power series of a path parameter a, in the vicinity of a known solution $\mathbf{V}_0$. Let $\mathbf{V}_1$ be a tangent vector at $\mathbf{V}_0$ and let a be defined as:

$$a := (\mathbf{V} - \mathbf{V}_0)^t \mathbf{V}_1 \tag{4}$$

Then $\mathbf{V}$ is developed to the order n (n=15 or 20 in applications):

$$\mathbf{V}(a) = \mathbf{V}_0 + a\mathbf{V}_1 + a^2\mathbf{V}_2 + \cdots + a^n\mathbf{V}_n \tag{5}$$

Operators are combined as follows:

$$\mathbf{C} = \mathbf{C}_0, \quad \mathbf{L}(\mathbf{V}) = \lambda \mathbf{C}_1 + \mathbf{L}_0 \mathbf{X}, \quad \mathbf{Q}(\mathbf{V}, \mathbf{V}) = \lambda \mathbf{L}_1 \mathbf{X} + \mathbf{Q}_0(\mathbf{X}, \mathbf{X}) \tag{6}$$

so Eq. (3) now reads as:

$$\mathbf{R}(\mathbf{V}) = \mathbf{Q}(\mathbf{V}, \mathbf{V}) + \mathbf{L}(\mathbf{V}) + \mathbf{C} \tag{7}$$

The series of Eq. (5) is substituted in Eq. (2), and powers of a are collected to obtain a family of linear systems on all $\mathbf{V}_i$:

- Order 0: $\mathbf{Q}(\mathbf{V}_0, \mathbf{V}_0) + \mathbf{L}(\mathbf{V}_0) + \mathbf{C} = 0$ which is true since $\mathbf{V}_0$ is a solution of Eq. (2).
- Order 1:

$$\mathbf{Q}(\mathbf{V}_0, \mathbf{V}_1) + \mathbf{Q}(\mathbf{V}_1, \mathbf{V}_0) + \mathbf{L}(\mathbf{V}_1) = 0 \tag{8a}$$

path parameter definition (4) leads to $\mathbf{V}_1^t \mathbf{V}_1 = 1$ (8b)