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Non-planar vibrations of a string in the presence of a boundary obstacle

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ABSTRACT

We analyze planar and non-planar motions of a string vibrating against a unilateral curved obstacle. Our model incorporates the change in tension due to stretching of the string, which introduces nonlinear coupling between motions in mutually perpendicular directions, as well as the wrapping nonlinearity due to the presence of the obstacle. The system of equations has been discretized by assuming functional form of the displacements which satisfies all the geometrical boundary conditions. This discretized system is then used to investigate the various motions possible both in the absence as well as the presence of the obstacle. In the absence of the obstacle, there are infinitely many planar and two non-planar motions viz. a circular trajectory and a precessing elliptical trajectory for a fixed magnitude of the disturbance. In contrast, the string has only one planar motion when the obstacle is present and two non-planar motions, either an oscillating orbit or a whirling orbit depending on the magnitude of the initial disturbance. To obtain the transition from oscillating to whirling orbits, we perform a stability analysis of the planar motion using Floquet theory. This analysis reveals that there exists a critical amplitude below which the planar motion is neutrally stable and the typical trajectories are ellipses with major and minor radii changing both in magnitude and direction. Beyond the critical amplitude, the planar motion is unstable and we get whirling trajectories which are precessing ellipses again with varying major and minor radii. We further study the effect of changing obstacle parameters on the critical amplitude, and obtain the stability boundaries in the space spanned by the obstacle parameters and the amplitude of the planar vibration. We obtain some interesting values of the obstacle parameters for which small and large amplitude planar motions are stable resulting in oscillating ellipses while motions with intermediate amplitudes are unstable giving precessing ellipses as the non-planar motion. We also find parameters for which the planar motion is always stable and hence, whirling motions are not possible. Finally, we consider non-planar vibrations with inclusion of several modes and observe more complicated non-planar motions due to modal interactions.

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1. Introduction

In this paper, we study the various possible motions of a string free to vibrate in two mutually perpendicular planes in the presence of a finite unilateral curved obstacle. This scenario is very relevant to the string vibrations in Indian stringed musical instruments like sitar and veena [1,2] which have a finite curved bridge at one end as shown in Fig. 1. These instruments are primarily excited parallel to the bridge but it has been observed that the strings have non-planar vibrations involving vibrations perpendicular to the bridge surface as well. Previous studies on non-planar motions of a string [3–11] have ignored the obstacle while studies on string vibrations in the presence of an obstacle [12–19] have considered only planar motions. We intend to fill this gap in the literature through this work.

Strings are the simplest continuous systems which exhibit vibrations. Strings find application in several fields, e.g., in cranes to lift loads and transmit power, in ropeways to suspend heavy weights, in elevators as a support system, in musical instruments etc. and they necessarily experience vibrations during operation. The mathematical model for small vibrations of a tightly stretched (with tension T) string is given by (see [20,21])

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad c^2 = \frac{T}{\rho}, \tag{1}$$

where “tightly stretched” is understood as constant tension, and ρ denotes the constant linear mass density. Exact solutions for the above linear partial differential equation can be obtained for simple boundary conditions. The real complication arises in situations where either the transverse motions are large enough that the assumption of small vibrations no longer holds or the supports are not ideal giving rise to non-trivial boundary conditions. Interestingly, real practical applications of string vibrations often take us into this realm. A meticulous examination of the practical problem of string vibrations reveals that the motion of the string is not against point ends but over distributed end supports (pulleys for cranes, ropeways and elevators, while bridge for the musical instruments) around which it wraps and unwraps during its vibrations. As a result, the effective length of the vibrating string for motions perpendicular to the support surface keeps changing with time leading to a moving boundary problem for the vibrations in this direction. For non-planar motions, it has been amply established in [3–11] that the assumption of a constant tension doest not correctly capture the interaction between motions in the perpendicular directions. Since the premise for (1) is not accurate, any conclusion drawn by solving this equation can not be relied upon from practical perspective. Thus, the challenge is in deriving the governing equation under calculated assumptions, writing the boundary conditions and realizing the crux of the problem using some appropriate technique which we have attempted in this paper.

Non-planar coupled motions of a string and the problem of string vibrating against a unilateral obstacle have been studied from time to time. Initial studies of this problem were restricted to experiments. Probably one of the first of these studies was by C.V. Raman [2] who made the observation that those overtones in case of tanpura or veena, Indian musical instruments, can be heard which according to the Young-Helmholtz law should have been absent. Later, sundry experimental studies [22–30] on the acoustics of Indian stringed musical instruments especially sitar and tanpura also illustrated the violation of Young-Helmhotz law and reported that several overtones were dominant for a long time. These studies clearly emphasized the importance of the presence of the finite-sized curved bridge in these instruments around which the string wraps and unwraps during its motion, thereby changing the effective length as well as the location of the nodes and antinodes [18,19]. This explains the inapplicability of the Young-Yelmholtz law. Invariably, the string vibrations in these experiments had non-planar motions as well as the participation of several modes. All previous studies on non-planar motions [3–11] have restricted their analysis to one mode in the two perpendicular directions. Accordingly, we also perform most of our studies using a single-mode approximation for vibrations parallel and perpendicular to the obstacle surface. However, towards the end, we briefly present some numerical results with more number of modes.

The assumption of constant tension even during vibrations can be justified for small amplitude vibrations in the string. However, this assumption is not always true in practical situations where the string undergoes large disturbances. In such

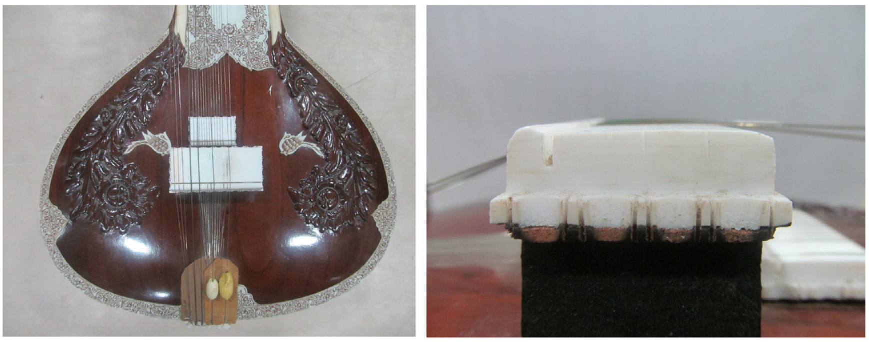

Fig. 1. Depiction of moving boundary in sitar, an Indian musical instrument.