Journal of Sound and Vibration 386 (2017) 327–335

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Effects of quadratic and cubic nonlinearities on a perfectly tuned parametric amplifier

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A R T I C L E I N F O

Article history:
Received 16 September 2015
Received in revised form
28 August 2016
Accepted 9 September 2016
Handling Editor: Ivana Kovacic
Available online 29 September 2016

A B S T R A C T

We consider the performance of a parametric amplifier with perfect tuning (two-to-one ratio between the parametric and direct excitation frequencies) and quadratic and cubic nonlinearities. A forced Duffing–Mathieu equation with appended quadratic nonlinearity is considered as the model system, and approximate analytical steady-state solutions and corresponding stabilities are obtained by the method of varying amplitudes. Some general effects of pure quadratic, and mixed quadratic and cubic nonlinearities on parametric amplification are shown. In particular, the effects of mixed quadratic and cubic nonlinearities may generate additional amplitude–frequency solutions. In this case an increased response and a more phase sensitive amplitude (phase between excitation frequencies) is obtained, as compared to the case with either pure quadratic or cubic nonlinearity. Furthermore, jumps and bi-stability in the amplitude–phase characteristics are predicted, supporting previously reported experimental observations.

1. Introduction

Parametrically amplifying (adding parametric to direct excitation for boosting resonant oscillations) microelectromechanical resonators, which in recent years have been used for filtering and sensoring [1,2], can be advantageous for signal amplification [3], and appear promising for energy harvesting [4,5]. They can conveniently be modelled with an appended cubic nonlinearity [6,7], reflecting the symmetric effects of nonlinear curvature or midplane stretching [8], with the nonlinear effects being comparably stronger due to the small length scale [9]. The effects of pure cubic nonlinearity for a parametric amplifier have been investigated in [10].

The effect of mixed quadratic and cubic nonlinearities is considered in the present work for two reasons. First, the quadratic nonlinearity can conveniently be introduced alongside the cubic nonlinearity as a correction term of the mathematical model. In this way it appears in the governing equation of motion as small compared to the cubic term. Secondly, the study of relatively strong quadratic nonlinearity is also relevant because it can model an asymmetry in restoring forces of elastic structures [11,12], e.g., due to buckling or initial curvature. The quadratic nonlinearity may even overcome the cubic nonlinearity, if the static deflection is large, or when the beam is very slender [13]. Therefore this study is motivated by an interest in general effects on parametric amplifiers, of both pure quadratic nonlinearity, and mixed quadratic and cubic nonlinearities.

Several works report on combined parametric and direct excitation including quadratic and cubic nonlinearities [14–16]. Commonly a perturbation method is applied, assuming damping, nonlinear, and excitation terms to be small, and that subthreshold (response dominated by the direct excitation component) pumping (adding parametric excitation) is applied.

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http://dx.doi.org/10.1016/j.jsv.2016.09.013
0022-460X/

In some cases it is also assumed that the quadratic nonlinearity is smaller than the cubic nonlinearity, e.g., [17]. The present work considers both when the quadratic nonlinearity is smaller and larger than the cubic nonlinearity, and focuses on superthreshold pumping (parametric instability threshold associated with an Arnold tongue). Superthreshold pumping is of interest because of the potentially higher achievable gains [10], as compared to being operated below their linear instability threshold, i.e. for subthreshold pumping.

The present study provides essential insights into the effects of mixed quadratic and cubic nonlinearities on parametric amplifiers. For instance it reveals a change in the number of amplitude–frequency solutions due to nonlinear effects and explains previously observed experimental jumps and bi-stability in the amplitude–phase characteristics [18].

In Section 2 a model system is proposed and the corresponding approximate analytical steady-state vibration amplitude is solved for using the method of varying amplitudes (MVA), for the cases of pure as well as mixed cubic and quadratic nonlinearities. In Section 3 these approximate analytical results are compared with results of direct numerical integration, showing good agreement. In Section 4 main conclusions of the paper are outlined.

2. Steady-state response analysis

A forced Duffing–Mathieu equation with unit-normalized linear natural frequency and additional quadratic nonlinearity is investigated:

$$\ddot{x}+\beta\dot{x}+(1+p\cos(2\Omega t))x+k_2x^2+k_3x^3=d\cos(\Omega t+\phi), \tag{1}$$

where $(\dot{\ })$ denotes temporal derivatives, $\beta=2\zeta$ where ζ is the damping ratio, k_2 is a quadratic nonlinearity coefficient, k_3 is a cubic nonlinearity coefficient, $(\ldots)x$ describes the linear elastic restoring force, p is a parametric excitation amplitude, d is a direct excitation amplitude, t is the time, and ϕ is the phase between the external and parametric excitation. A similar system but without quadratic nonlinearity has been investigated in [10]. It is the simplest one degree of freedom system which captures the effects of linear damping, quadratic and cubic nonlinearities, and which has both direct and parametric excitation, which are necessary for a parametric amplifier. Such a system is also physically easy to realize approximately [19–22].

To solve the steady-state oscillations $x(t)$ of (1) approximately, the MVA is employed, as proposed in [23] (confer also, e.g., [24]), where one assumes a harmonic series solution form:

$$x(t)=\sum_{m=0}^{n}A_{m1}(t)\cos(m\Omega t)+A_{m2}(t)\sin(m\Omega t), \tag{2}$$

where the amplitudes A_{m1} and A_{m2} are time-varying, and not required to vary slowly. This is contrary to the method of harmonic balance, where the coefficients A_{m1} and A_{m2} would be constants and (2) an approximation. The allowed time dependency of A_{m1} and A_{m2} means that (2) merely represents a shift of variables, by which the solution form (2) is exact for any value of n. The transition from x to $2n+1$ new variables, A_{m1} and A_{m2} ($A_{02}=0$), implies that a total of $2n+1$ equations are needed. Inserting (2) into (1) and requiring the coefficients of the involved harmonic terms to vanish identically, we introduce $2n$ additional equations; equation $2n+1$ then includes the remaining harmonic terms, including those having order higher than n. Considering $n=2$ and thus the first three harmonics in (2), yields:

$$x(t)=A_{01}(t)+A_{11}(t)\cos(\Omega t)+A_{12}(t)\sin(\Omega t)+A_{21}(t)\cos(2\Omega t)+A_{22}(t)\sin(2\Omega t), \tag{3}$$

and one obtains the following five equations with five variables A_{11}, A_{12}, A_{21}, A_{22}, and A_{01} to solve for:

$$\begin{aligned}\ddot{A}_{11}+\beta\dot{A}_{11}+2\Omega\dot{A}_{12}+\beta\Omega A_{12}+A_{11}\left(1-\Omega^2+\frac{1}{2}p+2k_2A_{01}+3k_3A_{01}^2\right)+(3k_3A_{01}+k_2)(A_{11}A_{21}+A_{12}A_{22})\\+\frac{3}{2}k_3A_{11}\left(\frac{1}{2}\left(A_{11}^2+A_{12}^2\right)+A_{21}^2+A_{22}^2\right)=d\cos(\phi),\end{aligned} \tag{4}$$

$$\begin{aligned}\ddot{A}_{12}+\beta\dot{A}_{12}-2\Omega\dot{A}_{11}-\beta\Omega A_{11}+A_{12}\left(1-\Omega^2-\frac{1}{2}p+2k_2A_{01}+3k_3A_{01}^2\right)+(3k_3A_{01}+k_2)(A_{11}A_{22}-A_{12}A_{21})\\+\frac{3}{2}k_3A_{12}\left(\frac{1}{2}\left(A_{11}^2+A_{12}^2\right)+A_{21}^2+A_{22}^2\right)=-d\sin(\phi),\end{aligned} \tag{5}$$

$$\begin{aligned}\ddot{A}_{21}+\beta\dot{A}_{21}+4\Omega\dot{A}_{22}+A_{21}+pA_{01}+2\Omega(\beta A_{22}-2\Omega A_{21})\\+\frac{1}{2}k_2\left(A_{11}^2-A_{21}^2+4A_{01}A_{21}\right)+3k_3A_{21}\left(\frac{1}{4}\left(A_{21}^2+A_{22}^2\right)+\frac{1}{2}\left(A_{11}^2+A_{12}^2\right)+A_{01}^2\right)+\frac{3}{2}k_3A_{01}\left(A_{11}^2-A_{12}^2\right)=0,\end{aligned} \tag{6}$$

$$\begin{aligned}\ddot{A}_{22}+\beta\dot{A}_{22}-4\Omega\dot{A}_{21}-2\beta\Omega A_{21}+A_{11}A_{12}(k_2+3k_3A_{01})+A_{22}\left(1-4\Omega^2+2k_2A_{01}\right)\\+3k_3A_{22}\left(\frac{1}{4}\left(A_{21}^2+A_{22}^2\right)+\frac{1}{2}\left(A_{11}^2+A_{12}^2\right)+A_{01}^2\right)=0,\end{aligned} \tag{7}$$