Sufficient and Necessary Conditions of Optimal Solutions of Semi-Infinite Convex Programing

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***Abstract*—This paper investigates characterizations of semi-infinite convex programming, and obtains conditions of optimal solutions of semi-infinite convex programming by using saddle point criteria.**

Keywords*—*semi-infinite programming; saddle point criteria; support function; support hyerplane, epigraph

I. INTRODUCTION

In this paper, we discuss the semi-infinite programming problem:

$$(\mathrm{SP})\begin{cases}\inf\{f(x)|x\in D\},\\ D:=\{x\in\Re^p|g_i(x)\le 0, i\in I\},\end{cases}$$

where f, $g_i\,(i\in I)$ are all real-valued functions in real n-dimensional vector space $\Re^p$, I is an infinite set (not necessarily a countable set). When f, $g_i\,(i\in I)$ are all convex functions, (SP) is called semi-infinite convex programming. Next, we mainly discuss this problem.

Let $P=\prod_{i\in I}\Re_i$ be the product space of $\Re_i(\cong\Re)$, the elements of P are $\Lambda=(\lambda_i)_{i\in I}$, $\lambda_i\in\Re$, $i\in I$. We denoted the direct sum space of $\Re_i(\cong\Re)$ by $S=\bigoplus_{i\in I}\Re_i$, the elements of S are $\Lambda=(\lambda_i)_{i\in I}$, of S are $\Lambda=(\lambda_i)_{i\in I}$, $\lambda_i\in\Re$ and $i\in I$, where $\lambda_i=0$ except for some finite numbers of λ_i. S is a proper subspace of P. $P_+(S_+)$ means the set of the elements $\Lambda=(\lambda_i)_{i\in I}$ with nonnegative coordinates in $P(S)$, which forms the convex cone in P and S respectively. We define in P or S: $(\lambda_i)_{i\in I}\le(\mu_i)_{i\in I}$ if and only if $\lambda_i\le\mu_i$. $P^*(S^*)$ means the algebraic conjugate space of $P(S)$, i.e., the set of the linear functionals in $P(S)$. Let

$$P_+^*:=\{\Lambda^*\in P^*|\langle\Lambda^*,\Lambda\rangle\ge 0,\forall\Lambda\in P_+\},$$
$$S_+^*:=\{\Lambda^*\in S^*|\langle\Lambda^*,\Lambda\rangle\ge 0,\forall\Lambda\in S_+\}.$$

P_+^* and S_+^* form the convex cone in P^* and S^* respectively. The Lagrange function of problem (SP) is

$$L:\Re^p\times S_+\to\Re,\ L(x,M)=f(x)+\sum_{i\in I}\mu_i g_i(x),\quad (1)$$

where $M=(\mu_i)_{i\in I}\in S_+$. Since numbers of $\mu_i\ne 0$ are finite, the sum in (1) can be achieved.

Definition1.1: Suppose $\bar{x}$ is an optimal solution of (SP). If there exists $\overline{M}\in S_+$, such that $(\bar{x},\ \overline{M})$ is a saddle point of L, i.e., for any $x\in\Re^p$ and $M\in S_+$, there exist inequalities for saddle points

$$L(\bar{x},M)\le L(\bar{x},\overline{M})\le L(x,\overline{M}),\ (2)$$

then we say that the problem (SP) satisfies the saddle point criterion at $\bar{x}$, or the problem (SP) is (Lagrange) regular at the point of $\bar{x}$.

Papers [1-6] described further studies of some conditions for regularity of the problem (SP) at optimal solutions. Shizheng [7] described a Lagrange condition for the regularity of the problem (SP) at optimal solutions by using the method of subdifferentials, and described further characterizations of regularity of (SP) by using the directional derivative. In this paper, we will introduce the concepts of regularity of feasible solutions and constraint functions, and give characterization of the regularities for the problem (SP).

II. REGULARITY FOR THE PROBLEM (SP)

Lemma 2.1[4]: Let $\bar{x}$ be an optimal solution of (SP), then the saddle point criterion holds at $\bar{x}$ (i.e., there exists $\overline{M}=(\bar{\mu}_i)_{i\in I}\in S_+$, such that (2) holds) if and only if for any $x\in\Re^p$, the following inequality holds

$$f(\bar{x})\le f(x)+\sum_{i\in I}\bar{\mu}_i g_i(x).\qquad (3)$$

We introduce the following sets and functions:

$$V := \{\Lambda = (\lambda_i)_{i\in I} \in P \mid \exists x \in \Re^p, g(x) \le \Lambda\}, \quad (4)$$

$$\varphi(\Lambda) := \inf\{f(x) \mid g(x) \le \Lambda, x \in \Re^p\}, \ \Lambda \in V. \quad (5)$$

Lemma 2.2[5]: Suppose the problem (SP) has a feasible solution, then $P_+ \subseteq V$, V is a convex set, and $\varphi(\Lambda)$ is a convex function.

Lemma 2.3[6]: 1) For any $x \in \Re^p$, then $g(x) \in V$, and $f(x) \ge \varphi(g(x))$;

2) Let $\bar{x}$ be an optimal solution of (SP), then $\varphi(0) = f(\bar{x}) = \varphi(g(x))$;

3) Given $\Lambda, \Lambda' \in V$, and $\Lambda \le \Lambda'$, then $\varphi(\Lambda) \ge \varphi(\Lambda')$;

4) For any $\Lambda \in V$, then $\partial\varphi(\Lambda) \subseteq -S_+$. Here the subdifferential of φ at Λ is a set

$$\partial\varphi(\Lambda) = \left\{ M = (\mu_i)_{i\in I} \in S \,\middle|\, \sum_{i\in I} \mu_i \lambda_i' \le \varphi(\Lambda') - \varphi(\Lambda), \forall \Lambda' = (\lambda_i)_{i\in I} \in V \right\}. \quad (6)$$

Lemma 2.4[7]: Let $\bar{x}$ be an optimal solution of the problem (SP), then there exists $\overline{M} \in S_+$, such that $(\bar{x}, \overline{M})$ is a saddle point of *L* (i.e., the problem (SP) regular at $\bar{x}$) if and only if $\partial\varphi(0) \ne \phi$.

For the convenience of discussion, this paper introduces the following sets:

$$S(\Lambda) := \{x \in \Re^p \mid g(x) \le \Lambda\}, \Lambda \in V;$$

$$W := \{(t, \Lambda) \in \Re \times V \mid \varphi(\Lambda) \le t\}.$$

W is the epigraph of the function $\varphi(\Lambda)$.

Lemma 2.5[8]: W is a convex set, $\alpha = (\varphi(0), 0)$ is a boundary point of W.

Definition 2.1: If the problem (SP) is regular at an optimal solution, then the problem (SP) is called regular.

We will try to characterize geometrically the regularity of the problem (SP). We know that there exists a support hyperplane at a boundary point of a convex set, i.e., there exists a support hyperplane which passes through the boundary point, and the convex set is on one side of the support hyperplane.

Theorem 2.1: Let $\bar{x}$ be an optimal solution of the problem (SP), then the problem (SP) is regular if and only if there exists a nonvertical support hyper plane of convex set W

$$\pi := \left\{ (t, \Lambda) \,\middle|\, vt + \sum_{i\in I} \mu_i \lambda_i = a, \ (t, \Lambda) \in \Re \times P, \ \Lambda = \{\lambda_i\}_{i\in I} \right\}$$

π is through $(\varphi(0), 0)$ in $\Re \times P$, where

$$M = (\mu_i)_{i\in I} \in S, v, a \in \Re \text{ and } v \ne 0.$$

Note: If the hyperplane π with the coefficient $v \ne 0$, then the hyperplane π is called nonvertical. Otherwise, the hyperplane π is called vertical.

Proof: If (SP) is regular, and $\bar{x}$ is an optimal solution of (SP), by Lemma 2.4, $\partial\varphi(0) \ne \phi$, i.e., there exists $\overline{M} \in \partial\varphi(0)$. Let $\overline{M} = \{\bar{\mu}_i\}_{i\in I}$, then by (6) the definition of $\partial\varphi(0)$, for any $\Lambda = (\lambda_i)_{i\in I} \in V$, we have

$$\sum_{i\in I} \bar{\mu}_i \lambda_i \le \varphi(\Lambda) - \varphi(0).$$

Take the hyperplane

$$\pi = \left\{ (t, \Lambda) \,\middle|\, t - \sum_{i\in I} \bar{\mu}_i \lambda_i = \varphi(0) \right\}.$$

For any $(t, \Lambda) \in W$, by the definition of W, then $\varphi(\Lambda) \le t$. Therefore,

$$t - \sum_{i\in I} \bar{\mu}_i \lambda_i \ge \varphi(\Lambda) - \sum_{i\in I} \bar{\mu}_i \lambda_i \ge \varphi(0).$$

Hence W is on one side of π.

On the other hand, let $t = \varphi(0)$, $\Lambda = 0$, then π is obviously through the point $(\varphi(0), 0)$. Thus π is a support hyperplane of the convex set W, which is through $(\varphi(0), 0)$. And $v = 1 \ne 0$. So the hyperplane is non-vertical.

Conversely, suppose that there exists a non-vertical support hyperplane of convex set W

$$\pi := \left\{ (t, \Lambda) \,\middle|\, vt + \sum_{i\in I} \mu_i \lambda_i = a \right\}, v \in \Re \text{ and } v \ne 0.$$

which is through $(\varphi(0), 0)$. Since π is through the point $(\varphi(0), 0)$, and $v\varphi(0) = a$. Then the equation of π can be rewritten as

$$vt + \sum_{i\in I} \mu_i \lambda_i = v\varphi(0).$$

Then W is on one side of π. Let us suppose if $(t, \Lambda) \in W$, then

$$vt + \sum_{i\in I} \mu_i \lambda_i \ge v\varphi(0). \quad (7)$$

For any $\Lambda \in V$, then $(\varphi(\Lambda), \Lambda) \in W$. Substitute (7), we get

$$-\sum_{i\in I} \mu_i \lambda_i \le v(\varphi(\Lambda) - \varphi(0)).$$

by Lemma 2.5 and the definition of W, $(\varphi(0) + 1, 0) \in W$. By substitute (7), we get

$$v \cdot (\varphi(0) + 1) + 0 \ge v \cdot \varphi(0) \text{ and } v \ge 0.$$