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Wave propagation with energy diffusion in a fractal solid and its fractional image

Gerd Gudehus [a,*], Asterios Touplikiotis [b]

[a] Emeritus, Institute of Soil and Rock Mechanics, Karlsruhe Institute of Technology, Germany
[b] Karlsruhe, Germany

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ABSTRACT

Crucial features of seismograms and spectra with small amplitudes are explained by means of fractality and fractional calculus. Wave propagations in the elastic range of porous solids imply precursors and followers of coherent waves. They result from a non-local diffraction via force chains which is called energy diffusion. Such phenomena are captured by fractional wave equations which are deduced by means of an elastic energy and the balance of momentum for random fractal ensembles. Theoretical propagations imply precursors which were similarly observed with bender elements, and a rate of dissipation nearly proportional to the kinetic energy which suits to resonant column test results. A novel three-dimensional fractional Dirichlet-Green function implies primary and secondary wave crests with speed and alignment which do not depend on the fractal dimension. Power spectra in the dislocation-free far-field of seismogeneous chain reactions and impacts tend to a fractality-dependent power law with a peak-like cutoff, both theoretically and observed, therein a modified Huygen's principle is employed. Limitations are discussed and possible extensions are indicated.

1. Introduction

Harmonic excitations of sand and fissured rock lead to higher harmonics with a kind of dispersive scattering during the propagation. We will explain such and related phenomena with a *linear theory*, whereas Nikolaev [1] proposed three kinds of nonlinear effects: a) stress-dependence of elastic parameters, b) anelastic behavior, c) triggered seismic activity. We focus on small deviations from equilibria with uniform static stress fields so that (a) means incremental linearity with stress-dependent compliances. Except for near-fields considered at the ends of Sections 2, 4 and 5 we leave aside dislocations during propagations, thus (b) is ruled out. This requires small amplitudes and a stable elastic range, which excludes also (c).

Sand bodies and rock formations indicate stochastic (random) fractality, but there is as yet no consent upon how to capture it mathematically. Fractality of gases and liquids arises and changes with critical phenomena at or off thermodynamic equilibria, respectively. We are dealing with *fractal solids* which exhibit self-similar roughness already at and near stable equilibria. Even in a fractal sense they are often not homogeneous, which means multi-fractality (Mandelbrot [2,3]). Fractal dimensions have been determined for coastlines and rock surfaces by means of box counting, but cannot likewise be identified for fault zones as cracks are entangled so that internal surfaces are not uniquely given. Thus the 'band-limited fractal random medium' proposed by Wu and Aki [4] for wave scattering is rather subjective. Shapiro and Faizullin [5] proposed 'fractals of a turbulence medium' and 'discrete random scatterers', this is disputable as they relate likewise calculated power spectra with fractal dimensions of cracks.

In these two and similar publications the field of propagation velocities is assumed to vary spatially in a fractal manner. E.g., Man et al. [6] propose $Q^{-1} \propto w^{\gamma}$ with angular frequency ω and $-1 < \gamma \leq -0.5$ for the *quality factor* Q, and relate their γ with an ambiguous fractal dimension of cracks. Combining a scattering velocity model with a classical wave equation, Van der Baan [7] derives a frequency-independent Q beyond a cutoff. Lithosphere data do not suffice to verify one or the other Q-factor. Resonant column tests with dry sand indicate that its rate of dissipation is proportional to the kinetic energy in an elastic range (e.g. Huber [8]). Hardin [9] derived a constant damping ratio $D \equiv 1/Q$ by means of a viscosity which is proportional to the excitation frequency. However, dry sand is not viscous and a constitutive relation should not contain an excitation frequency. Thus Q-factors are questionable for fissured rock and sand.

Following Prof. Rudolf Gorenflo, we propose the notion *energy diffusion* for the diffraction in a fractal solid. Its mechanism, outlined in Section 2 by means of force chains, differs from classical

* Corresponding author.

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ones because of the self-similar roughness. It implies a kind of damping in the elastic range, i.e. without dislocations. Other than with fluids in the laminar regime, it is *non-local* as propagation paths of different sizes and directions are entangled. Other than the diffusion of heat in a resting solid, the diffusion of seismic energy occurs during propagations. We use this notion in oder to avoid common connotations of 'damping', 'diffraction', 'scattering' and 'dispersion'.

Combinations of a classical wave equation with a fractal velocity model cannot represent the balance of momentum for fractal solids as terms with gradients of stiffness (e.g. Vrettos [10]) are left aside and classically impossible with self-similar roughness. Instead, *fractional wave equations* have been derived from the balance of momentum with fractality (Gudehus and Touplikiotis [11], Gorenflo et al. [12]), therein an elastic energy in its stable range is employed which depends on the elastic strain. Confining to small deviations from equilibrium, vectorial integro-differential equations are obtained which can be reduced to algebraic equations (Section 3). This 'fractional image' represents expectation values, but its interpretation, validation and calibration are admittedly uncommon. Sand bodies are advantageous because they can repeatedly be assembled as samples or deposits, whereas fractal rock samples and formations are hardly reproducible.

We evaluate fractional wave equations for Dirac-like and for temporarily harmonic boundary conditions (Section 4). Linear operators are employed and legitimate for small elastic deviations from equilibria. The features of energy diffusion indicated above are obtained as temporal and spatial fractional derivatives are non-instantaneous and non-local, respectively (Gorenflo and Mainardi [13]). Employing one- and three-dimensional fundamental solutions, we derive properties of power spectra for far-field propagations which suit to observations (Section 5). Three main conclusions, viz.

1) Fractality-independent wave crests and alignment, but fractality-dependent fast precursors and slow followers.
2) A rate of dissipation nearly proportional to the kinetic energy for not too big space-time sections.
3) A trend of power spectra towards a fractality-dependent power-law with a peak-like cutoff.

are discussed, and desirable extensions are indicated (Section 6). The relation of random fractality and its fractional image for porous solids is outlined in the Appendix.

2. Propagation and energy diffusion in the elastic range of porous solids

Observations indicate that sand and fissured rock have a *fractal pore system*. πόϱος means passage, so pores are voids among grains or fissures of rock forming a permeable system. Other than with self-similarly rough lines and surfaces, Mandelbrot's [2] mass fractal is adequate for three dimensions: The solid mass in a control cube of length d is $m = m_r(d/d_r)^{3\alpha}$ with reference values m_r, d_r and a fractal dimension α in the range ca. $0.9 < \alpha < 1$. The latter suits to the experience that Monge's and Delesse's rule, stating equality of volume and surface mass fractions, works fairly well although it implies $\alpha = 1$, and that sand deposits exhibit density fluctuations over several orders of magnitude. Fractal pore systems ensue also from nested shear band patterns (e.g. Gudehus [14]). Such fractals are stochastic so that m denotes the expectation value of a random ensemble, and require cutoffs: control cubes are meaningless when being smaller than the size of grains or rock fractions, and also beyond the size of a considered region.

We idealize sand and fissured rock as *fractal solids* with such a distribution of their mass. With a single mineral the solid mass in a control cube is the number of enclosed mineral molecules multiplied by the mass of each one. Likewise extensive is the mass of pore fluid, the energy (with kinetic, gravitational, elastic, chemical and thermal parts), and the linear momentum. Straight lines traversing a fractal solid would exhibit self-similarly rough distributions of mass, energy and momentum with a fractal dimension α. We focus on fractally homogeneous (or uniform) solids for which control cubes with different centre positions have the same m_r and α. Sand and fissured rock are rather multi-fractal, i.e. their α is site- and time-dependent, but let us first see how a single α with the proposed narrow range works.

Except for the ends of Sections 2, 4 and 5 we consider fractal solids in a *stable elastic range*. There is a specific elastic energy w_e, depending on elastic strain components ϵ_{ij}^e, which is the potential of equilibrium stress components σ_{ij} via $\sigma_{ij}=\partial w_e/\epsilon_{ij}^e$ (Jiang and Liu [15]). In the case of stability small deviations of $w_e\epsilon_{ij}^e$ from equilibrium values can be approximated by a quadratic form which is positively definite as otherwise kinetic energy would arise. This implies a σ_{ij} in the convex range of w_e and requires solid bridges between grains or rock fractions. The elastic energy in a control cube of length d is $E_e=w_e d^3(d/d_r)^{3\alpha}$ with d_r and α as for m, which means that the molecules have elastic strain and stress with the same spatial fractality as their mass. ϵ_{ij}^e and σ_{ij} are state variables which cannot be related with displacement gradients and force densities, respectively, in the same way as without fractality.

A fractally distributed equilibrium stress field is transmitted by *force chains* via contact flats of grains or rock fractions in a jammed fabric. These were observed with photoelastic discs (e.g. Fig. 1 from [16]) and obtained with numerical simulations (e.g. Radjai [17]). They arise also in jammed fabrics of natural grains or rock fractions, and in entangled clusters of them up to big sizes, thus geotechnical and tectonic stress fields indicate self-similar roughness (Gudehus [14], Heidbach et al. [18]). Momentum pulses propagate along branching force chains, the transmitting contact flats being wider for bigger contact forces, with directions and speeds which scatter fractally around size-dependent spatial averages. Thus coherent waves induce faster precursors and slower followers, while smaller incoherent waves are generated by force chains. The balance of linear momentum cannot be homogenized to a classical wave equation for such propagations.

The implied *energy diffusion* is a mechanism in its own right. It has little in common with the scattering of light or sound in calm air as it results from the entangled non-locality of the momentum

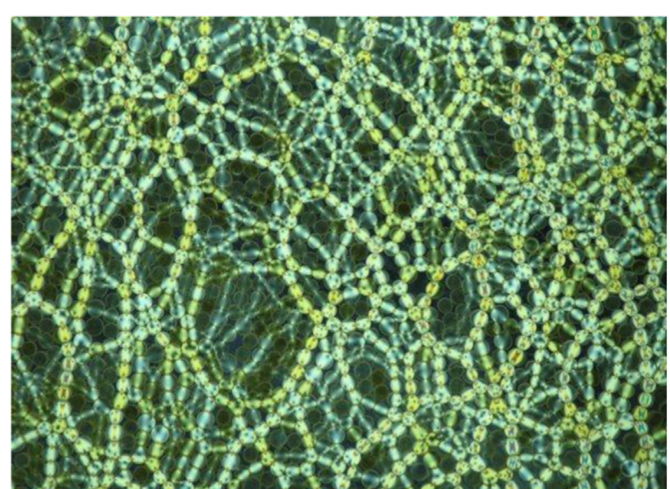

Fig. 1. Force chains in a jammed fabric of photoelastic discs (from Behringer et al. [18]).