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An adaptive directional importance sampling method for structural reliability analysis

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ABSTRACT

The importance sampling is merged with directional simulation in this paper. A sampling function is defined on the unit hyper sphere which samples random directions. The directions are sampled around a direction that aims to the design point. The sampling function uses spherical coordinates to generate random directions. The method is made adaptive by a closed form updating rule to renew the sampling parameters. To reduce the number of calls on the limit state function, a root finding procedure is put forward. The proposed method is tested with well-known test problems and its performance is compared with the conventional directional simulation. The results demonstrate the accuracy and efficiency of the proposed method for rare event estimation.

1. Introduction

Engineering systems involve uncertainties which need to be considered in order to have a realistic design and analysis. Reliability theory provides methods to address this need [1–7]. The reliability assessment in the field of engineering can be encapsulated in the integral defined as follows [8]:

$$P_f(G(\mathbf{x}) \leqslant 0) = \int_{G(\mathbf{x})\leqslant 0} f(\mathbf{x})d\mathbf{x} \tag{1}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ represents random variable vector, including loads, material properties, and modeling uncertainties. Moreover, $G(\mathbf{x})$ and $f(\mathbf{x})$ denote the limit state function and joint probability density function, respectively. The integral is evaluated in the failure domain where $G(\mathbf{x}) \leq 0$. The limit state function returns negative values when the failure occurs and positive, provided that the system performs safely; accordingly, this function divides the stochastic domain into safety and failure regions. The boundary of the two regions is called the limit state surface as Fig. 1 depicts. Practically, the above mentioned integral cannot be evaluated directly because the limit state function is not usually presented explicitly; additionally, high-dimensional practical problems make it impossible to have an analytical evaluation of the integral. The numerical evaluation also turns to be more difficult when the shape of the failure region becomes more complicated. As a result, various approximate [9–13], response surface [14–16], and simulation methods [17–20] have been presented thus far.

The first-order-second-moment reliability method (FORM) and second-order reliability method (SORM) [21] are among the first attempts to approximate the failure probability. FORM and SORM approximate the limit state function with first-order and incomplete second-order functions, respectively. These methods require the solution of an optimization problem to find the design point and its distance from the origin. FORM and SORM are fairly efficient with respect to simple and mild nonlinear limit states, but are completely inaccurate in the case of limit states with multiple design points or high nonlinearity [22].

Despite approximate methods such as FORM, simulation methods can provide the results with arbitrary precision at the cost of more computational efforts. An accurate estimation of the failure probability of a structure generally requires a large number of Monte Carlo simulations and limit state evaluations. Since often the limit states are not explicitly presented, a costly numerical method such as FEM is usually utilized [9]. To resolve this problem, several methods have been presented. The aim of variance reduction techniques is to reduce the statistical fluctuations and yield more accurate results with fewer computational efforts. A number of simulation techniques broadly applied are importance sampling, line sampling, directional simulation and subset simulation [18,23,24].

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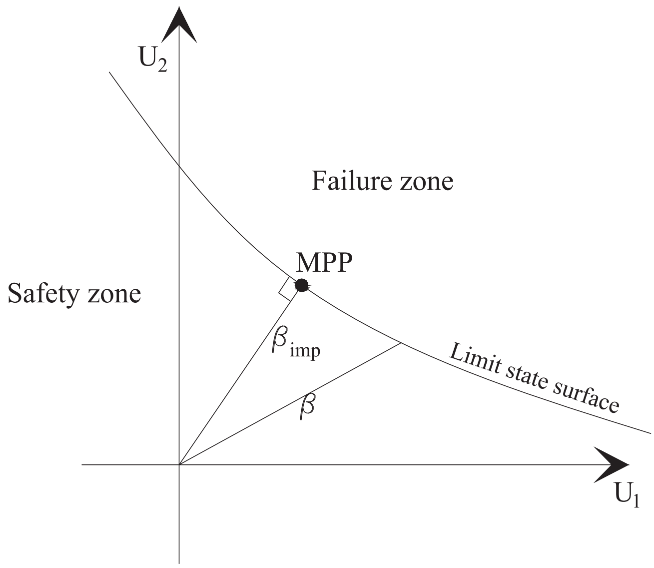

Fig. 1. Definition of β in the polar integration equation.

The central idea in importance sampling technique is sampling in the most important parts of the stochastic space which have higher probability content; therefore, more accurate probability estimations with less variance could be gained [25–27]. Directional simulation samples polar directions in standard normal space. The probability content for each direction is evaluated by a one-dimensional integration which has a closed-form solution in the standard normal space [28]. Directional simulation and importance directional simulation have been widely studied by Ditlevsen et al. [29,30] and Melchers [31], and Moarefzadeh and Melchers [32]. In a paper by Bjerager [33] some directional sampling densities are presented for particular classes of the limit state functions. Nie and Ellingwood [28] presented three deterministic procedures to sample the uniformly distributed directions. These methods lack the adaptability with respect to the limit state and become inefficient for highly-nonlinear limit states or low-failure probabilities. Subsequently, Nie and Ellingwood [34,35] made an adaptive directional sampling technique which applies a neural network in order to distinguish important regions on the unit hyper sphere and utilizes a finer mesh for sampling on those regions. Grooteman [36] developed an adaptive radius-based importance sampling procedure and, in another paper by him [22], a surrogate-based adaptive directional importance sampling.

This study is aimed at developing an adaptive directional importance sampling. To do this, inspired by the cross entropy-based adaptive importance sampling method presented by [9], a new importance sampling procedure is introduced into the directional simulation scheme. An outline of the proposed method is demonstrated in the sequel. A number of well-known example limit states are employed to compare the proposed method with other procedures and prove its efficiency.

A set of non-normal interdependent random variables can always be transformed into independent standard normal variables by applying appropriate transformations such as Nataf [37] or Rosenblatt transformations [38]. Thus, the remainder of the paper is dedicated to independent standard normal space, usually called U-space. Before introducing the new directional importance sampling method, a brief scheme of directional simulation is presented.

The rest of paper is structured as follows. Sections 2 and 3 discuss a brief overview of directional simulation and cross entropy-based importance sampling, respectively. The details of the proposed method are demonstrated in Section 4. In Section 5, the accuracy and efficiency of the proposed method are examined by test problems, and the results are compared with crude Monte Carlo (MCS) and Directional Simulation (DS). Finally, a conclusion is put forward in Section 6.

2. Directional importance sampling

The failure probability in the n-dimensional standard-normal space can be expressed by the following integral [39,40]:

$$P_f = \int_S \int_\beta (2\pi)^{-n/2} \exp\left(-\frac{r^2}{2}\right) r^{n-1} dr ds \tag{2}$$

The inner integral is taken along the radial direction, and the outer integral is taken on the unit hypersphere surface denoted by S. Each radius coincides with the limit state surface in a point at the distance of β from the origin (See Fig. 1).

The inner integral can be expressed as a chi-squared cumulative distribution function; therefore, the integral (2) would be read as follows [40]:

$$P_f = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_S [1-\chi_n^2(\beta^2)] ds \tag{3}$$

where $\chi_n^2()$ denotes the cumulative chi-squared distribution function with n degrees of freedom, and $\Gamma(\cdot)$ is the gamma function. By introducing the importance directional sampling function $q(.)$, the Eq. (3) is expressed as:

$$P_f = \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_S \frac{[1-\chi_n^2(\beta^2)]}{q(\boldsymbol{\alpha})} q(\boldsymbol{\alpha}) ds \tag{4}$$

The directional sampling function $q(.)$ is a density function defined on the surface of the unit hypersphere and randomly samples the points on the unit n-sphere or equally speaking unit vectors $\boldsymbol{\alpha}$. Regarding the Eq. (4), the estimation for the failure reliability is:

$$\hat{P}_f = \frac{1}{N} \frac{\Gamma(n/2)}{2\pi^{n/2}} \sum_{i=1}^{N} \frac{[1-\chi_n^2(\beta_i^2)]}{q(\boldsymbol{\alpha_i})} \tag{5}$$

where the unit vectors $\boldsymbol{\alpha_i}$ are the samples based on the distribution $q(.)$. If the directional sampler is uniform on the hypersphere, the denominator of Eq. (5) is canceled out with the coefficient $\frac{\Gamma(n/2)}{2\pi^{n/2}}$, and the Eq. (5) is reduced to:

$$\hat{P}_f = \frac{1}{N} \sum_{i=1}^{N} [1-\chi_n^2(\beta_i^2)] \tag{6}$$

which is the unbiased estimator of conventional directional simulation [22].

It must be noted that obtaining the β_i values requires finding the coincidence point of the sampled directions $\boldsymbol{\alpha_i}$ with the limit state surface. There are very efficient root finding algorithms such as the secant method which is utilized here.

3. Cross entropy-based adaptive importance sampling

There are several variance reduction methods, one of the most popular of which is parametric importance sampling. This technique exploits a proper sampling density function $q(\mathbf{x},\mathbf{v})$ with the parameter vector $\mathbf{v}$ to estimate the integral $\int h(\mathbf{x})d\mathbf{x}$ as follows [9]:

$$\int h(\mathbf{x}) d\mathbf{x} = \int \frac{h(\mathbf{x})}{q(\mathbf{x},\mathbf{v})} q(\mathbf{x},\mathbf{v}) d\mathbf{x} \tag{7}$$

$$\int h(\mathbf{x}) d\mathbf{x} \approx \frac{1}{N} \sum_{i=1}^{N} \frac{h(\mathbf{x}_i)}{q(\mathbf{x}_i,\mathbf{v})} \tag{8}$$