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A new unbiased metamodel method for efficient reliability analysis

Guofeng Xue [a,b], Hongzhe Dai [a,b,*], Hao Zhang [c], Wei Wang [a,b]

[a] School of Civil Engineering, Harbin Institute of Technology, Harbin 150090, China
[b] Key Lab of Structures Dynamic Behavior and Control (Harbin Institute of Technology), Ministry of Education, Harbin 150090, China
[c] School of Civil Engineering, The University of Sydney, NSW 2006, Australia

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ABSTRACT

Metamodel method is widely used in structural reliability analysis. A main limitation of this method is that it is difficult or even impossible to quantify the model uncertainty caused by the metamodel approximation. This paper develops an improved metamodel method which is unbiased and highly efficient. The new method formulates a probability of failure as a product of a metamodel-based probability of failure and a correction term, which accounts for the approximation error due to metamodel approximation. The correction term is constructed and estimated using the Markov chain simulation. An iterative scheme is further developed to adaptively improve the accuracy of the metamodel and the associated correction term. The accuracy and efficiency of the new metamodel method is illustrated and compared with the classical Kriging metamodel and high dimensional model representation methods using a number of numerical and structural examples.

1. Introduction

A common technique for evaluating structural reliabilities with complex limit state functions is to use the metamodel method. It uses a strategic design of experiments (DoE) to obtain an analytical approximation of the relationships between the input random variables and the limit state response of interest. Earlier application of this approach is the use of the response surface methods [1]. Construction of metamodels is a challenging problem. Recent developments include but not limited to artificial neural networks [2–4], support vector machines [5–8], high dimensional model representation (HDMR) [9,10], polynomial chaos expansion [11,12] and Kriging [13,14]. For the commonly used polynomial-based metamodel, the results may be sensitive to the selected interpolation polynomials and their parameters due to the rigid and non-adaptive structure of the polynomials [6]. For instance, although polynomial chaos can be used for local interpolation, the definitions of the design of numerical experiments and of the polynomial degrees are tricky [11]. The performance of artificial neural networks cannot be guaranteed due to the fitting problems as there is no efficient constructive method for choosing the structure and the learning parameters of artificial neural network [5]. In addition to these limitations, a general drawback of the metamodel method is that it is difficult or even impossible to quantify the error caused by approximating the actual limit state function (LSF) by a metamodel [15–17].

In order to overcome the aforementioned difficulties, this paper develops a new efficient metamodel method which is unbiased. The basic idea is to formulate an unknown probability of failure as the product of a metamodel-based failure probability and a correction term, which accounts for the approximation error due to metamodel approximation. Although this idea is mathematically straightforward and has been used in structural reliability analysis very recently [18,17], the construction and the estimation of the correction term is a challenging task in such methods. For instance, the correction term in [18] is constructed as the sum of two terms, one involves the union of the actual and the metamodel-based failure regions, and the other involves the intersection of such regions. As a result, two different sets of samples from both union and intersection are required to estimate the correction term. This may decrease the efficiency for estimating the correction term. Although the line sampling is used to accelerate the estimation of the correction term, the determination of the important direction in line sampling is a nontrivial task. In this paper, the correction term is constructed by introducing an intermediate event, which is the union of the actual failure region and the metamodel-based failure region. Since only the samples from the union of the actual and the metamodel-based failure region are required, as compared with [18], the correction term can be estimated more efficiently using the Markov chain simulation. To further improve the efficiency of the proposed method, an adaptive

* Corresponding author at: School of Civil Engineering, Harbin Institute of Technology, Harbin 150090, China.
E-mail address: hzdai@hit.edu.cn (H. Dai).

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refinement procedure is developed to simultaneously improve the metamodel and the corresponding correction term. It shall be noted that the hybrid use of variance reduction-based simulation methods and metamodel method has been proposed more recently, e.g., [16,19,20]. The metamodel in such methods are employed in the context of simulation methods to estimate the response of the samples. These methods are generally considered as the improvement of the underlying simulation-based methods. For example, in [19] a Kriging metamodel is combined with an importance sampling that makes use of Markov chains. The Kriging metamodel is used to predict the response of important samples, as one step of the importance sampling. Overall, the reliability analysis method is simulation-based, as opposed to the metamodel method considered in this paper.

The paper is organized as follows: the Kriging metamodel is briefly introduced in Section 2, followed by the presentation of the proposed unbiased metamodel method in Section 3. The procedure of the proposed method is then summarized in Section 4. Four examples are then given to demonstrate the application and efficiency of the proposed method, including a finite element analysis (FEA)-based reliability assessment in which the limit state function has to be evaluated implicitly through FEA. Comparisons of the proposed method and the conventional metamodel methods, including Kriging metamodelling and high dimensional model representation, are made.

2. Kriging method

Among the available metamodel methods, herein we focus on the Kriging method, which has gained significant attention in the field of structural reliability theory in recent years [13,14,21]. It should be noted that the proposed method of constructing and estimating the correction term is general and can be applied to any metamodel method, and not restricted to the Kriging metamodel discussed here. This section briefly introduces the Kriging method for the completeness of introducing the proposed methodology. Details about Kriging method can be found elsewhere, e.g, [22,23].

Kriging metamodel is an interpolation technique based on statistical theory, which consists of a parametric linear regression model and a nonparametric stochastic process [22]. It requires DoE to determine its stochastic parameters and then predictions of the response can be computed on any unknown sample. Given an initial DoE $\mathbf{X} = [\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(p)}]^{\mathrm{T}}$, with $\mathbf{x}^{(i)} \in \mathbb{R}^n$ $(i = 1, \ldots, p)$ the ith input, and $\mathbf{Y} = [g(\mathbf{x}^{(1)}), \ldots, g(\mathbf{x}^{(p)})]^{\mathrm{T}}$ with $g(\mathbf{x}^{(i)}) \in \mathbb{R}$ the corresponding response to $\mathbf{x}^{(i)}$. The approximate relationship between any sample $\mathbf{x}$ and the response $g(\mathbf{x})$ can be denoted as

$$g(\mathbf{x}) = F(\boldsymbol{\beta}, \mathbf{x}) + z(\mathbf{x}) = \mathbf{f}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\beta} + z(\mathbf{x}) \tag{1}$$

where $\mathbf{f}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\beta}$ is the regression model representing the trend of the model, which is defined by a set of basis functions $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})]^{\mathrm{T}}$ and the corresponding regression coefficients $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_m]^{\mathrm{T}}$. In the ordinary Kriging, $F(\boldsymbol{\beta}, \mathbf{x})$ is a scalar and always taken as $F(\boldsymbol{\beta}, \mathbf{x}) = \beta$. So the estimated $g(\mathbf{x})$ can be simplified as

$$g(\mathbf{x}) = \beta + z(\mathbf{x}). \tag{2}$$

Here $z(\mathbf{x})$ is a zero-mean stationary Gaussian process with autocovariance at points $\mathbf{x}$ and $\mathbf{w}$ defined as

$$\mathrm{cov}(z(\mathbf{x}), z(\mathbf{w})) = \sigma^2 R(\mathbf{x}, \mathbf{w}) \tag{3}$$

where cov = covariance, σ^2 is the process variance and $R(\mathbf{x}, \mathbf{w})$ is the autocorrelation function. The most widely used autocorrelation function is anisotropic Gaussian model and is adopted in this paper:

$$R(\mathbf{x}, \mathbf{w}) = \exp\left(-\sum_{i=1}^{n} \theta_i (x_i, w_i)^2\right) \tag{4}$$

where x_i and w_i are the ith component of the points $\mathbf{x}$ and $\mathbf{w}$ respectively, and θ_i is the correlation parameter in the ith dimension.

Define $\mathbf{R}$ as a $p \times p$ symmetric correlation matrix with $\mathbf{R}_{ij} = R(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}),\ i,j = 1, \ldots, p$, and $\mathbf{F}$ as a $p \times 1$ unit vector, then β and σ^2 are estimated as

$$\hat{\beta} = \left(\mathbf{F}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{F}\right)^{-1}\mathbf{F}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{Y}, \tag{5}$$

$$\hat{\sigma}^2 = \frac{1}{p}(\mathbf{Y} - \mathbf{F}\boldsymbol{\beta})^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{Y} - \mathbf{F}\boldsymbol{\beta}). \tag{6}$$

The correlation parameter θ can be obtained through the maximum likelihood estimation:

$$\boldsymbol{\theta} = \arg\min_{\boldsymbol{\theta}} (\det \mathbf{R})^{\frac{1}{p}} \hat{\sigma}^2. \tag{7}$$

Since there exists corresponding interpolation model for each θ, the best Kriging model can be obtained by optimizing θ.

Then at an unknown point $\mathbf{x}^{(0)}$, the Best Linear Unbiased Predictor (BLUP) of the response $\tilde{g}(\mathbf{x}^{(0)})$ and Kriging variance $\sigma^2_{\tilde{g}}(\mathbf{x}^{(0)})$ are computed as

$$\tilde{g}(\mathbf{x}^{(0)}) = \mathbf{f}^{\mathrm{T}}(\mathbf{x}^{(0)})\boldsymbol{\beta} + \mathbf{r}(\mathbf{x}^{(0)})^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{Y} - \mathbf{F}\boldsymbol{\beta}), \tag{8}$$

$$\sigma^2_{\tilde{g}}(\mathbf{x}^{(0)}) = \hat{\sigma}^2\left(1 + u(\mathbf{x}^{(0)})^{\mathrm{T}}\left(\mathbf{F}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{F}\right)^{-1}u(\mathbf{x}^{(0)}) - \mathbf{r}(\mathbf{x}^{(0)})^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{r}(\mathbf{x}^{(0)})\right) \tag{9}$$

where $\mathbf{r}(\mathbf{x}^{(0)}) = [R(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}), \ldots, R(\mathbf{x}^{(0)}, \mathbf{x}^{(p)})]^{\mathrm{T}}$ and $u(\mathbf{x}^{(0)}) = \mathbf{F}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{r}(\mathbf{x}^{(0)}) - 1$.

3. An unbiased metamodel method

Although some metamodels like Kriging can provide a measure of the local uncertainty of the prediction of new samples, i.e., Kriging variance, the overall error resulting from replacing the actual LSF with the metamodel cannot be quantified. This model uncertainty is the epistemic uncertainty of the metamodel. It cannot be quantified by the metamodel itself. As a consequence, the direct use of Kriging metamodel will inevitably result in a biased estimator of the probability of failure. Having identified this issue, we propose a correction term to quantify the bias of the metamodel-based failure probability, and formulate the unknown probability of failure as a product of the metamodel-based failure probability and a correction term. In this manner, the bias of the metamodel-based failure probability can be accounted for and an unbiased estimator of the failure probability is obtained.

Let $\tilde{g}(\mathbf{x})$ be a Kriging metamodel for the real LSF $g(\mathbf{x})$, and $\widetilde{F} = \{\mathbf{x} | \tilde{g}(\mathbf{x}) \leqslant 0\}$ be the metamodel-based failure region for the real failure region $F = \{\mathbf{x} | g(\mathbf{x}) \leqslant 0\}$. The correction term, denoted by K, is defined as

$$K = \frac{P(F)}{P(\widetilde{F})} \tag{10}$$

where $P(F)$ and $P(\widetilde{F})$ is the failure probability and the metamodel-based failure probability, respectively. Then $P(F)$ can be written as

$$P(F) = K \cdot P(\widetilde{F}). \tag{11}$$

Eq. (10) shows that the correction term K quantifies the error resulting from substituting $g(\mathbf{x})$ with $\tilde{g}(\mathbf{x})$, thus it can be used to consider the bias of the metamodel-based failure probability $P(\widetilde{F})$ even a poor metamodel $\tilde{g}(\mathbf{x})$ is employed. By multiplying $P(\widetilde{F})$ with K, an unbiased estimator of $P(F)$ is achieved as shown in Eq. (11).

Clearly, the key issue of the method is the computation of the correction term K. Since one cannot guarantee that the