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Multivariate log-concave probability density class for structural reliability applications

Farzad Faridafshin [a,*], Arvid Naess [b]

[a] Sustainable Arctic Marine and Coastal Technology (SAMCoT), Centre for Research-based Innovation (CRI), Norwegian University of Science and Technology, 7491 Trondheim, Norway
[b] Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway

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A B S T R A C T

Structural reliability analysis is conventionally based on a description of uncertainty via a joint probability density function (JPDF). This paper builds on an alternative concept of working with a probability distribution class, which is the set of all distributions that satisfy several prior pieces of information. A multivariate probability class is introduced given the first- and second-moment information and the condition on log-concavity of the JPDF, which is versatile enough to cover the majority of multivariate probabilistic models that are typically used in reliability applications. Owing to the strong mathematical properties of this class, it is shown that a reliability analysis in the multidimensional space of uncertainty is reduced to a univariate problem, given the linearity of the failure surface with respect to uncertain parameters. Therefore, a generalization of the Chebyshev inequality for the univariate class of distributions with a log-concave PDF is applied to calculate the upper bound of the probability of failure. The benefit of this method is that fitting a JPDF, particularly with limited amounts of data, is facilitated, yet the method provides a tight but not overly pessimistic estimate of the probability of failure. A bivariate numerical example is provided for demonstration.

1. Introduction

Structural reliability analysis is conventionally based on a description of uncertainty via a joint probability density function (JPDF). Methods, such as the method of moments or maximum likelihood, are often used to estimate the parameters of an assumed or justified probability model based on a set of data points. In a multivariate setting, in addition to the choice of probabilistic model, the correlations among the uncertain parameters also need to be determined. Different techniques exist for establishing the JPDF based on a dataset, such as postulating a known multivariate parametric model whose parameters should be estimated, establishing the pairwise dependence of random variables by using known bivariate probabilistic models, copula modeling, or the conditional modeling approach; see, e.g., [1–3]. The probability of failure is then calculated as the total probability mass that lies in a so-called failure region, as described by a set of performance functions or safety margins.

The present paper builds on an alternative concept of working with a probability distribution class. A distribution class is the set of all probability distributions that satisfy a certain number of prior pieces of information as constraints, with some directly based on data and some by assumption. Such prior information can, for instance, be distributional moment information, support of the distribution, or properties such as symmetry or unimodality of the JPDF. Reliability analysis then becomes a matter of calculating the minimum reliability or its complement, the maximum probability of failure, given any probability distribution belonging to this set (often called a distributional set). The same methodology can be applied when reliability-based design optimization is the focus, in which case design parameters need to be chosen such that the structure maintains a minimum level of reliability. The higher-level problem of reliability-based design is formulated in this paper, which also covers the lower-level problem of reliability analysis. In this paper, this methodology is called a “distributionally robust” framework, as it is commonly referred to by the optimization community in relation to “distributionally robust optimization”; see, e.g., [4–7]. In civil and mechanical engineering communities, this concept is categorized under the theory of “imprecise probabilities.” see, e.g., [8,9].

* Corresponding author at: Vestre Raa 50F, 5239 Raadal, Norway.
E-mail addresses: Farzad.Farid@ntnu.no (F. Faridafshin), Arvid.Naess@ntnu.no (A. Naess).

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The background for the proposed methodology is the situations with "little data", and in particular "little multidimensional data". As noted in [10], which describes the motivation behind this research in more detail, "in situations with little data, it is difficult to 'standardize' a method for determining the best possible JPDF based on conventional uncertainty analysis methods; an expert always has to be involved, and the experts might not be consistent. The motivation was therefore to find out if there is something that can make life easier by avoiding arguing about distribution types and parameter fitting when faced with limited data."

A multivariate class of distributions is introduced in this article, which builds on the first two distributional moments as well as a so-called log-concavity constraint for the JPDF as the prior information. This class can evidently also be applied to univariate settings or to scenarios with the independence of multiple uncertain parameters. The use of this class offers some major benefits:

1. One does not need to assume or establish any specific probability density functions. Establishing the probability density function (PDF) might be specifically challenging or somewhat subjective in multivariate situations—i.e., when a JPDF is involved—particularly when the number of available data points is not large. Instead, here, one only postulates that the JPDF has the specific property of being logarithmically concave. This implies a relevant constraint on the shape and tail behavior of the true probability distribution. As will be seen in this paper, the assumption of log-concavity has the power of incorporating a large number of parametric probability distributions commonly used in structural reliability analysis and might be a natural choice in many situations.
2. One only needs to extract the vector of means and the covariance matrix from a multidimensional dataset. This is a simple objective exercise, which may allow for working with datasets that do not contain a significant number of data points. Still, since the first- and second- moment information are combined with a realistic assumption on the shape of the distribution, i.e., log-concavity, the proposed probability bound is considerably tighter than the bound obtained from the application of the Chebyshev or unimodal classes of distributions. For the definition of the Chebyshev and unimodal classes, see, e.g., [10,11] (these classes are also briefly discussed in Section 3.3).

In this paper, a log-concave distribution refers to a distribution whose JPDF is log-concave. Therefore, we are concerned about a log-concave density class (or a log-concave PDF/JPDF class), even though we simply call this class a "log-concave class." This is in line with the majority of references available on the topic; see, e.g., [12–16]. It is also possible to define a class of distributions based on the log-concavity of the cumulative distribution function (CDF); see, e.g., [17,18]. The definition of log-concavity based on the density function allows the exploitation of the multivariate properties of the log-concave densities, which is not possible if log-concavity is defined based on the CDF. Such a multivariate performance of the log-concave densities provides the background for this article. This is despite the fact that the log-concave CDF class is slightly larger than the log-concave density class and covers few more parametric distribution types; see, e.g., [18] for a list of parametric distributions belonging to each class.

The paper is organized into five sections. Section 2 introduces the concept of log-concavity and its properties before the main result is presented in Section 3. Section 4 presents an application example of the main result, and Section 5 concludes the paper.

2. Log-concavity and its properties

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leqslant \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}) \quad \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n \text{ and } \forall \lambda \in [0,1], \tag{1}$$

and a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is concave if $-f$ is convex. Graphically, a function is convex if its epigraph (the area above the curve/surface) is a convex geometry—i.e., a convex set of points—and a function is concave if its hypograph (the area below the curve/surface) is a convex geometry; see Fig. 1. The epigraph and hypograph of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ are defined as

$$\begin{aligned} \text{epi} f &= \{(\mathbf{x}, t) \in \mathbb{R}^{n+1} : f(\mathbf{x}) \leqslant t\} \\ \text{hyp} f &= \{(\mathbf{x}, t) \in \mathbb{R}^{n+1} : f(\mathbf{x}) \geqslant t\}. \end{aligned} \tag{2}$$

The concept of log-concavity is used in this paper. This concept is directly defined for a probability density function (PDF) instead of any arbitrary function, even though the definition would be very similar for the arbitrary case. Here, the notation $\mathbf{u}$ is used for the vector of uncertain parameters (or the random vector).

A multivariate probability density function $f : \mathbb{R}^{nu} \rightarrow [0,\infty)$ is log-concave if it can be expressed as the exponent of a concave function, i.e., $f = \exp(\varphi(\mathbf{u}))$, where $\varphi : \mathbb{R}^{nu} \rightarrow (-\infty,\infty)$ is concave. An example is the multivariate normal density, where $\varphi(\mathbf{u})$ is a concave quadratic in $\mathbf{u}$, and therefore log-concave.

Equivalently, $f : \mathbb{R}^{nu} \rightarrow [0,\infty)$ is log-concave if $\log f$ is a concave function—i.e., if

$$\begin{aligned} &\log f(\lambda\mathbf{u}_1 + (1-\lambda)\mathbf{u}_2) \geqslant \lambda \log(f(\mathbf{u}_1)) + (1-\lambda)\log(f(\mathbf{u}_2)) \\ &\quad \forall \lambda \in [0,1] \text{ and } \forall \mathbf{u}_1, \mathbf{u}_1 \in \mathbb{R}^{nu} \\ &\text{with convention } \log 0 = -\infty. \end{aligned} \tag{3}$$

Many commonly used univariate parametric distributions have log-concave densities. Examples include normal, Gumbel, exponential, logistic, Laplace, Rayleigh, uniform, Weibull (with a shape parameter greater than or equal to 1), gamma (with a shape parameter greater than or equal to 1), power function (with a parameter greater than or equal to 1), and beta distribution (with both parameters greater than or equal to 1). For a more complete list, see [18]. Several of these distributions are popular in structural reliability applications. Distributions that are not log-concave include lognormal and Pareto distributions. Several distributions are selected here, and their densities, together with the logarithms of their densities, are plotted in Fig. 2. As seen from the logarithmic plots, all the hypographs (except for lognormal) are convex regions, and thus the underlying PDFs are log-concave. The distribution functions plotted here are the results of fitting distributions to a set of data points based on the method of maximum likelihood. The dataset has a mean value of 3.7 and a standard deviation of 0.923 (see the numerical example). This paper is concerned with the much more general case of multivariate densities. Therefore, a

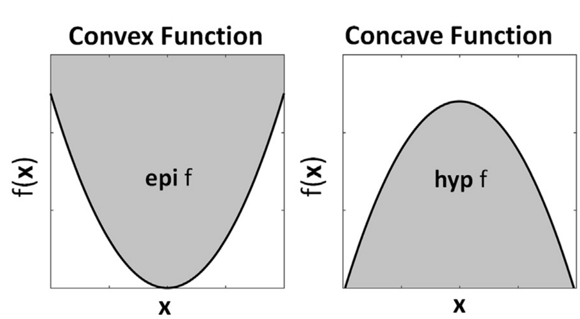

Fig. 1. Convex vs. concave function $f : \mathbb{R}^n \rightarrow \mathbb{R}$.