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Alternative Admissible Functions for Natural Frequencies and Modeshapes of a Beam with Lumped Attachments

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ABSTRACT

In this paper, the use of a combination of trigonometric and low order polynomials as admissible functions to be used with the method of assumed modes is investigated for the calculation of the natural frequencies and modeshapes of a beam with lumped attachments. Since the admissible functions do not satisfy the boundary conditions, penalty terms are used to replace the constraints of the boundary conditions of the beam, with virtual stiffness elements of appropriate values representing the boundary conditions. By comparison with previously obtained results, the proposed method using the assumed modes approach with admissible functions and penalty terms is evaluated for accuracy and computational effectiveness. It is shown that the proposed method is accurate and shows no ill-conditioning for the problem of an Euler-Bernoulli beam with lumped attached elements.

1. Introduction

The problem of determining the eigenvalues (natural frequencies) of a dynamical system has attracted the attention of many researchers. Several methods have been developed in the course of recent decades to consider this problem, for example assumed modes [1,2], Lagrange multipliers or the superposition method as developed by Gorman in [3]. Common to all methods is the necessity to choose a set of functions to represent the spatial domain in whichever method is chosen for the computation of the eigenvalues and modeshapes. Since the true eigenfunctions obtained through purely analytical approaches often involve functions such as hyperbolic functions whose behaviours are numerically unstable, efforts have been made to introduce alternative admissible functions which are numerically stable and do not display the ill-conditioning that can occur when the analytical eigenfunctions are used.

As representative examples, Bhat in [4] introduced Characteristic Orthogonal Polynomials (COP) to find the first six natural frequencies of a rectangular plate. He concluded that the mutual orthogonality of the admissible functions renders the mathematical procedure more straightforward due to the fact that the inner product of two orthogonal functions is zero. Many researchers followed Bhat's footstep in utilizing COP's. Ahmadi and Nikkhoo in [5] used BCOP's (Bhat's Characteristic Orthogonal Polynomials) to solve the forced vibration problem of a non-uniform Euler-Bernoulli beam. They employed the method of assumed modes, as well as the Gram-Schmidt orthogonalization process to solve for three different kinds of boundary conditions. However, they concluded that the orthogonality condition does not present a computational advantage. Accordingly, other researchers such as Brown and Stone in [6] also refuted or downplayed the significance of orthogonality as a simplifying factor, concluding that the accuracy of the results is a function of the degree of the basis original functions and not a property related to their orthogonality. Brown and Stone also suggested that the orthogonal polynomials provide little advantage over non-orthogonal polynomials when it comes to accuracy.

Other researchers have investigated the use of special polynomials to ensure numerical stability. Yuan and Dickinson employed an extended Rayleigh-Ritz approach that had previously been applied to beams to study dynamical systems consisting of rectangular plates [7]. The authors employed orthogonal polynomials developed by Bhat in [8] to generate their admissible functions. This extended Rayleigh-Ritz approach was applied to three cases: a stepped plate, box beams and plates with slits. They compared the results obtained for the symmetric and antisymmetric mode shapes for the first sixteen mode shapes of a box beam using Bhat Characteristic Orthogonal Polynomials against those of an exclusively sine Fourier series. Close agreement was observed.

The Rayleigh-Ritz and assumed shape methods typically require that the chosen functions satisfy the boundary conditions of the problems and work has been done to investigate if this requirement can be modified. For instance, Li used a combination of Fourier series and

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compensating polynomials to simulate the free vibrations of a beam with general boundary conditions [9]. The degree of the compensating polynomial was determined by the order of derivative which was needed. Based on the conclusion regarding the degree of the polynomial, this method was shown to converge quickly and the addition of even a single term to the Fourier series can make a great difference. The same author in [10] compared the convergence rate of sine and cosine Fourier series, offering an in-depth look into the method proposed in [9] by considering different kinds of boundary conditions. It was concluded that the nature of the boundary condition plays a role in the convergence rate of the series such that a simply-supported boundary condition is most suitably represented by a sine Fourier series while a cosine Fourier series simulates a clamped-clamped boundary condition more appropriately.

Work has also progressed along other lines of investigation, namely analysis into the effects of the choice of basis functions that do not necessarily satisfy the boundary conditions, in conjunction with penalty methods to implement the boundary conditions. This differs from a traditional approach where the basis functions are typically chosen to satisfy the boundary conditions. Monterrubio and Ilanko proposed a set of alternative admissible functions consisting of polynomials up to 2nd degree and cosine functions [2]. By only considering low degree polynomials, they avoided the numerical instabilities that are intrinsic to polynomials. They also took advantage of penalty functions to implement the boundary conditions. The authors laid out the proof for convergence of their set of proposed functions in [11]. Further work by Monterrubio in [12] used the Rayleigh-Ritz in combination with the penalty function approach to obtain 27 frequencies and buckling parameters of a complex geometric shape with simply supported boundary conditions.

The penalty function method for the implementation of boundary conditions or other constraints was comprehensively investigated in [13] where simultaneous use of stiffness and inertial penalty functions to simulate beams and plates (shells) with constraints was considered. Monterrubio and Ilanko argued the possibility of using both virtual masses as well as virtual springs to simulate the boundary conditions. They showed that stiffness penalty functions converge monotonically from below while inertial (mass) penalty functions converge from above. Subsequently, Ilanko and Dickinson investigated the effect of using negative stiffness penalty functions for beam boundary conditions on the convergence of the Rayleigh-Ritz approach [14]. Specifically, the effect of using negative stiffness values alongside positive stiffness values on the eigenvalues of a clamped-simply supported beam, as well as a circular stepped beam with simply supported boundary conditions at both ends was investigated.

As outlined in the brief literature review above, alternative choices of functions replacing analytical eigenfunctions have been investigated for the use of the calculation of natural frequencies of beams. However to the best of the authors' knowledge, the application of these alternative functions for the calculations of natural frequencies and modeshapes of beams with lumped attachments has not been investigated to date.

The majority of the research conducted on the problem of lumped attachments to beams has used the analytical eigenfunctions of the same system without lumped attachments as the choice of basis functions. For example, in [15] Cha applied the eigenfunctions of the bare beam to the case of a beam with either simply-supported or cantilever boundary conditions to which one or several lumped elements are attached. The mass and stiffness matrices of the bare beam were then modified by taking into account the values of the lumped elements, as well as the values of the eigenfunction vector at the point of attachment. In [16] the authors solved the fourth-order partial differential equation for the case of a cantilever beam to which a mass-spring system with an extra degree of freedom was attached in the middle, by applying the displacement compatibility at the point of attachment. In [17], the authors compared the analytical results for a beam loaded with a lumped mass, incorporating various boundary conditions, with the approximate results obtained using Rayleigh's expression. They concluded that although the closed form Rayleigh expressions are less exact, they are able to provide faster results. Along the same lines, the work in [18] considered the case of an added lumped mass attached to an Euler-Bernoulli beam with 16 different combinations of boundary conditions, namely clamped, pinned, sliding, and free, all using analytical eigenfunctions. Three values of point mass to beam mass ratios were considered, and the first three normalized natural frequencies and corresponding normalized mode shapes were obtained for four different mounting positions along the beam. In [19], Nicholson et al. took advantage of separation of variables combined with the Green's function to derive the exact natural frequencies and orthogonal mode shapes of a cantilever beam with a grounded mass-spring system, as well as a simply-supported beam with a suspended mass-spring system. They compared their exact results with the approximate Galerkin approach as well as the finite element approach. They concluded that the exact method has precedence over the approximate methods in terms of exactness and numerical economy. An older method of using Lagrange multipliers in determining the natural frequencies of combined dynamical systems is illustrated in [20], where Dowell solved the problem of a plate with braces using the kinetic and potential energies for the free plate. Lagrange multipliers were used to account for the presence of braces and constraints. The eigenvalues may then be obtained after substituting the expressions for kinetic, potential, and Lagrange multipliers into Lagrange's equations.

Although the use of alternative basis functions has been investigated and shown great promise for use with frequency analysis of beams and plates, the same methods have yet to be applied to the case of a beam with lumped attachments. In this paper, we combine the use of the set of functions developed by Monterrubio and Ilanko in [2] with the penalty function method to obtain the natural frequencies and modeshapes of an Euler-Bernoulli beam with either simply-supported or clamped-free boundary conditions. Then, the effects of adding lumped attachments on the effectiveness of the method are investigated and the findings are compared with previously published results in the literature.

2. Theory

2.1. Derivation of the equations of motion

The method of assumed modes is often used to discretize a dynamical system, much in the same manner as Finite Element Analysis (FEA). However, unlike FEA this method employs global elements rather than finite elements and superimposes a finite number of assumed modeshapes to simulate the vibrations of the dynamical system. This method is explained in full detail in [21].

In this paper, the method of assumed modes is used to model the transverse vibrations of an Euler-Bernoulli beam with attachments. The transverse displacements of the beam w, are assumed to be separable functions of space and time. Therefore, the transverse vibration of the beam $w(x,t)$ are assumed to be given by the mathematical form

$$w(x,t) = W(x)\cos(\omega t) \tag{1}$$

where x is the position along the beam, t represents time, ω is the frequency of vibration and W represents the amplitude of vibration. The space-dependent function, W, is assumed to be a sum of assumed modes and thus to have the form

$$W(x) = \sum_{i=1}^{N} c_i \phi_i(x) \tag{2}$$

Eq. (2) is then substituted into Rayleigh's quotient to determine the mass and stiffness matrices and thus solve for the eigenvalues (natural frequencies) and eigenvectors (modeshapes).