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Analysis of flexural behavior of a simple supported thin beam under concentrated multiloads using trigonometric series method

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ABSTRACT

For accurately predicting the local stresses around the loading points, a stress function in the form of trigonometric series (TS) was applied to obtain the displacements and stresses for simply supported thin beams under concentrated multiloads. The convergences of TS solutions were testified for both displacements and stresses, where the convergence criteria were established and the proper iteration numbers were given. Besides, the accuracy of TS solutions was verified by finite element analysis, where the stress concentration effect is obvious for the shear stresses around the loading sections, not following the parabolic distribution proposed in literatures. Finally, taking thin beams under multi-wheels loads as an example, parameter studies were then performed to examine the effects of wheels' numbers, distances and locations on the flexural response of beams. Numerical results were summarized into a series of curves indicating the distribution of displacements and stresses for various parameters.

1. Introduction

Simply supported thin beams subjected to concentrated loads are a class of mechanic problems generally encountered in practical engineering, such as the web of I-shaped girders or off-track box girders under wheel loads, as shown in Fig. 1. Unlike the uniform load, the stress concentration around the loading point is generally obvious, which possibly induces the local buckling before overall yielding, so that more attentions need to be paid in design.

Due to the practical importance, the concentrated loading case has been focused by many researchers since the 1903's. The stress of an infinite long beam under two equal and opposite concentrated loads was initially analyzed by Filon using Fourier series [1], and he found that the stresses around the loading point were remarkable and diminished rapidly with the increment of the distance from the loading point. However, the convergence property around the loading point is not clear and the proper iteration numbers are not clearly specified for stresses and displacements. Afterwards, Seewald [2] solved the problems on the infinite long beam loaded by a concentrated force based on the semi-infinite-plane theory, and found out that the vertical stress (normal to the axis) has a good agreement with the exact solution from the Fourier series solution [1], but the warping one (along the neutral axis) at the bottom edge of the beam has a large deviation up to 90.9%. Similarly, Wang [3] solved the warping stress of the deep beam subjected to a concentrated load by using the semi-infinite-plane theory and the superposition principle, however, the error for the warping stress at the top edge reaches to 26.3% in comparison with the finite element analysis (FEA) result. In FE analysis, gradually refined grid will lead to the large increment of local stress at the loading point, which well corresponds to the infinite stress at the loading point obtained from the Fourier series method [1] and the semi-infinite-plane theory [2,3].

According to the simplification of the deformation, many beam theories have been established, including the classical beam theory (CBT), the first-order beam theory (FBT), and the high-order beam theory (HBT). The CBT known as the Euler-Bernoulli beam [4] is only applicable to slender beams. While for thick or deep beams, the CBT underestimates deflection and overestimate natural frequency and buckling load due to the ignorance of the transverse shear deformation effect [5]. The FBT known as the Timoshenko beam theory [6,7] is proposed to overcome the drawback of CBT by accounting for the transverse shear deformation effect for deep beams. In FBT, a shear correction factor (SCF) is needed to compensate the discrepancy between the actual stress state and the assumed constant stress state since

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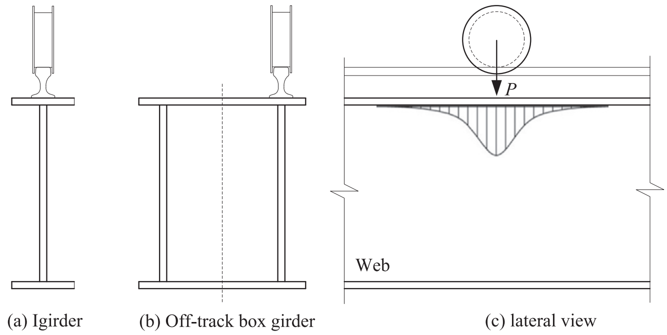

Fig. 1. Examples of thin beams (webs) under concentrated wheel loads.

the FBT violates the zero shear stress boundary conditions on beam edges [8]. The SCF depends on various parameters, such as geometry configurations, material properties, boundary conditions [9]; and, therefore, it needs a further research [10].

Besides, in order to avoid the use of SCF and get a better estimation of behavior of deep beams, the HBT, represented by various shape functions for the shear stress, has been developed, including the third-order theory [11], the trigonometric theory [12], the hyperbolic theory [13], the exponential theory [14,15], the mixed theory [16–19]. Although these HBTs were initially proposed for plates and shells, application of the shape functions to beams is immediate. Based on the assumption of a high-order variation of axial displacement and a constant transverse displacement, most of HBTs comply with the zero shear stress boundary conditions and produce a non-linear (generally parabolic-shaped) distribution for the transverse shear stress through the beam height. Applying a high-order variation to both axial and transversal displacements [17,18], Carrera [20–24] proposed the Carrera Unified Formulation (CUF). Then, Demasi [25] provides a hierarchical formulation leading to very accurate FE models for beams, plates and shells, in which the stretching effect is automatically taken into account.

A detailed observation on the literature reveals that the transverse shear strain generally varies in the form of parabolic function in most of HBTs. However, this parabolic variation may not be applicable to the shear strain around the loading sections for beams under concentrated loads, due to the stress concentration. Therefore, this paper deals with the flexural behavior of thin beams under concentrated multiloads by using the trigonometric series (TS) method. The emphasis is placed on the convergence property of TS solutions for both stresses and displacements, especially for those around the loading points, where two convergence criteria were established depending on the iteration type, and the proper iteration numbers were given. Also, the accuracy of TS solutions is verified by FEA for both displacements and stresses, especially for the transverse shear stresses around the loading sections, where the stress concentration is obvious, not following the parabolic-shaped distribution proposed in literatures. Finally, taking thin beams under multi-wheels loads in parameter study, the effects of wheels' numbers, distances and locations on the displacements and stresses are investigated respectively.

2. Trigonometric series method

A thin beam with rectangular cross section subjected to a concentrated load is investigated firstly in Fig. 2a. Due to the small variability of stresses within the beam thickness, the original 3D beam model can be simplified into a 2D one with a unit thickness, as shown in Fig. 2, and the load is correspondingly transferred to P/t. For analysis, the coordinate system O-yz is established in Fig. 2b with its original point O set on left top of the beam. The beam is made of a homogeneous, isotropic and linearly elastic material with Young's and shear moduli E and G, respectively. The span is l and the height is h. The

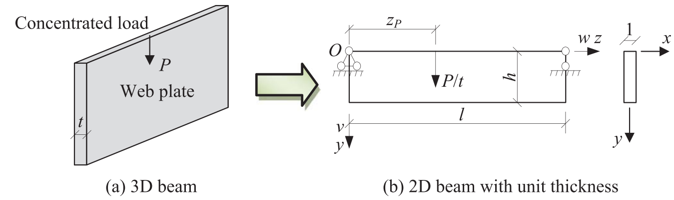

Fig. 2. Thin beam under single concentrated load.

concentrated load P/t is applied at z_P on the upper edge. The y-/z-axial displacements are v and w, respectively.

Considering the discontinuity for concentrated load, the stress function φ needs to be expanded in the form of trigonometric series (TS), given by

$$\varphi = \sum_{i=0}^{\infty} (A_i cos\alpha_i z + B_i sin\alpha_i z) f_i(y) \tag{1}$$

where i is the iteration number. $\alpha_i = i\pi/l$. A_i and B_i are the constant coefficients to be determined by boundary conditions.

Substituting Eq. (1) into the compatibility equation,

$$\frac{\partial^4\varphi}{\partial z^4} + 2\frac{\partial^4\varphi}{\partial z^2\partial y^2} + \frac{\partial^4\varphi}{\partial y^4} = 0 \tag{2}$$

a four-order differentiate equation is obtained,

$$f_i^{(4)}(y) - 2\alpha_i^2 f_i^{(2)}(y) + \alpha_i^4 f_i(y) = 0 \tag{3}$$

where the subscripts (2) and (4) indicate the second- and four-order differentiates over variable y, respectively. The solution then is

$$f_i(y) = c_{1i}\cosh\alpha_i y + c_{2i}\sinh\alpha_i y + c_{3i} y\cosh\alpha_i y + c_{4i} y\sinh\alpha_i y \tag{4}$$

where c_{ki} ($k = 1, 2, 3, 4$) are constant coefficients.

Consequently, the stress function φ is

$$\varphi = \sum_{i=0}^{\infty} (A_i cos\alpha_i z + B_i sin\alpha_i z)(c_{1i} cosh\alpha_i y + c_{2i} sinh\alpha_i y + c_{3i} y cosh\alpha_i y + c_{4i} y sinh\alpha_i y). \tag{5}$$

Based on the relations between the stress function φ and the stresses, we have

$$\sigma_z = \frac{\partial^2\varphi}{\partial y^2} = \sum_{i=1}^{\infty} \alpha_i^2 (A_i\cos\alpha_i z + B_i\sin\alpha_i z)\left[\left(c_{1i} + \frac{2c_{4i}}{\alpha_i}\right)\cosh\alpha_i y + \left(c_{2i} + \frac{2c_{3i}}{\alpha_i}\right)\sinh\alpha_i y + c_{3i} y\cosh\alpha_i y + c_{4i} y\sinh\alpha_i y\right] \tag{6}$$

$$\sigma_y = \frac{\partial^2\varphi}{\partial z^2} = -\sum_{i=1}^{\infty} \alpha_i^2 (A_i\cos\alpha_i z + B_i\sin\alpha_i z)[c_{1i}\cosh\alpha_i y + c_{2i}\sinh\alpha_i y + c_{3i} y\cosh\alpha_i y + c_{4i} y\sinh\alpha_i y] \tag{7}$$

$$\tau_{zy} = -\frac{\partial^2\varphi}{\partial z\partial y} = \sum_{i=1}^{\infty} \alpha_i^2 (A_i\sin\alpha_i z - B_i\cos\alpha_i z)\left[\left(c_{2i} + \frac{c_{3i}}{\alpha_i}\right)\cosh\alpha_i y + \left(c_{1i} + \frac{c_{4i}}{\alpha_i}\right)\sinh\alpha_i y + c_{4i} y\cosh\alpha_i y + c_{3i} y\sinh\alpha_i y\right] \tag{8}$$

In order to obtain the coefficients c_{ki}, the concentrated load P/t is also expanded in the form of TS. To do this, we assume that the load P/t is acted within a infinitesimal domain [$z_P - \varepsilon/2$, $z_P + \varepsilon/2$], given by

$$\frac{P}{t} = \frac{1}{t}\lim_{\varepsilon\to 0}\frac{P}{\varepsilon} = \frac{2P}{lt}\sum_{i=1}^{\infty}\sin\alpha_i z_P \sin\alpha_i z, \tag{9}$$

where ε is the size of domain. Substitute the Eqs. (6)–(8) into the stress boundary conditions,