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# The generalised constrained finite strip method for thin-walled members in shear

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ABSTRACT

The constrained finite strip method (*c*FSM) is an extension of the semi-analytical finite strip method (SAFSM) of structural analysis of thin-walled members, where consideration of the displacement fields utilised and of various mechanical criteria allows constraint matrices to be formed. The application of these constraint matrices to the linear buckling eigenvalue problem of the SAFSM results in deformation fields that satisfy the considered criteria and, therefore, isolate particular modes. Through careful selection of the mechanical criteria, the deformation fields obtained may be restricted to particular buckling modes. This is referred to as modal decomposition. While the *c*FSM has been applied to modal decomposition of thin-walled, prismatic members under the action of longitudinal normal stresses, it has yet to be applied to such members under the action of shear stresses. Recent work using the SAFSM to analyse the buckling behaviour of thin-walled, prismatic members under applied shear stresses, notably by Hancock and Pham, has shown that the issues of potentially indistinct minima or multiple minima in the signature curve can occur under this loading, as they did for compression and bending. This paper briefly presents the derivation of a SAFSM that permits coupling between longitudinal series terms of sines and cosines and also considers membrane instability due both to shear stresses and transverse normal stresses. It then presents the application of the *c*FSM to such a finite strip and results are produced for members under shear stresses. While the results are presented for members with unrestrained ends (equivalent to infinitely long members), simplification via removal of the degrees of freedom not present in typical FSM formulations would allow finite length members with simply-supported ends to be analysed.

## 1. Introduction

The finite strip method (FSM) developed by Cheung [1] is a specialisation of the finite element method that utilises longitudinal regularity of the analysed member to reduce the dimension of the problem being analysed. First utilised for local buckling analysis of thin-walled members by Przemienicki [2], before being extended to other forms of buckling by Plank and Wittrick [3], the FSM has become an indispensable design tool thanks to its ability to generate a curve showing the critical elastic buckling stress of a section as a function of the buckling half-wavelength, when only a single longitudinal half-wavelength is considered; this is known as the signature curve of a section and is a concept that was popularised by Hancock [4]. The ubiquity of the FSM in the analysis and design of thin-walled, cold-formed steel members has only become more prevalent with the development of the Direct Strength Method (DSM) [5] which, in practice, predicts the ultimate strength of a member by considering the signature curve and the geometric and material properties of the section. Typically, this process involves taking the buckling stress values of the signature curve at its two minima as the critical stresses corresponding to local and distortional buckling, and using these in the DSM strength equations [6,7]. However, there are many sections for which the signature curve may not have two minima, or may have more than one minimum for local or distortional buckling [8]. Further, the buckling modes at the lengths where the signature curve attains its minima may not be 'pure' local or distortional modes. These ambiguities in the signature curve prompted the development of the constrained finite strip method (*c*FSM) [9–11], which draws on the mechanical assumptions of Generalised Beam Theory (GBT) [12] in order to define pure global, distortional and local buckling modes. Application of the *c*FSM, available in the finite strip computer program CUFSM [13], then allows the critical buckling stresses of these pure local and distortional buckling modes to be determined. As the DSM is calibrated based on signature curves developed for the general (i.e. unconstrained) deformation field of the FSM, application of *c*FSM to the DSM and further development of the *c*FSM are fields of ongoing

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research [14–16].

Until recently, the DSM has only been applicable to members under longitudinal normal stresses, i.e. compression and/or bending; it has recently been extended to C-section members under shear [17]. As for the DSM for compression and bending, the DSM for shear requires knowledge of the signature curve of the section. The first semi-analytical finite strip capable of analysing members under shear stresses was that of Plank and Wittrick [3], which was recently revitalised by Hancock and Pham [18,19] who applied the formulation to the buckling of C-sections under shear stresses. It should be noted that, as their SAFSM assumes longitudinal regularity, including stresses, the moment gradient necessary for a section under shear to be in equilibrium cannot be replicated and so the members analysed are under 'pure' shear. Their analysis, as well as subsequent analyses, such as of C-sections with longitudinal stiffeners [20], revealed signature curves that display many of the same ambiguities of those developed for members in compression and/or bending; i.e. indistinct minima and possible mode coupling.

In light of this, applying the *c*FSM methodology to members in pure shear would prove useful, both as a theoretical tool for examining the buckling behaviour of such members and for assisting in further development of the DSM for shear. As the *c*FSM methodology is largely separate from the FSM to which it is applied, this paper will first present a finite strip that may be applied to members in a combined loading state, where all components of the Green-Lagrange in-plane strains are considered in formulating the stability matrices. Coupling between longitudinal series terms of different numbers of half-wavelengths is also permitted. Subsequently, the application of the *c*FSM methodology to this FSM will be elucidated, using the recently-developed generalised *c*FSM [21,22].

## 2. SAFSM for applied shear

### 2.1. Linear buckling analysis

The linear buckling eigenvalue problem of the SAFSM is a second-order analysis formulated via the theorem of stationary potential energy. The total potential energy of the system is the sum of the internal elastic strain energy $U_e$, which is obtained by evaluating the energy stored by the actions of the internal linear stresses $\boldsymbol{\sigma}_L$ in the linear strains $\boldsymbol{\varepsilon}_L$, and the potential energy due to the externally applied stresses, which is obtained as the negative of the work $W$ of the applied stresses $\boldsymbol{\sigma}$ in the respective non-linear strains $\boldsymbol{\varepsilon}_{NL}$. This total potential energy is as shown in Eq. (1), where the integral is over the volume of the elements of the system and the linear stresses and strains are related by $\boldsymbol{\sigma}_L = \mathbf{E}\boldsymbol{\varepsilon}_L$, where $\mathbf{E}$ is the relevant constitutive matrix for the problem at hand.

$$U_e - W = \int_V \left(\frac{1}{2}\boldsymbol{\sigma}_L^{\mathrm{T}}\boldsymbol{\varepsilon}_L - \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}_{NL}\right)dV = \int_V \left(\frac{1}{2}\boldsymbol{\varepsilon}_L^{\mathrm{T}}\mathbf{E}\boldsymbol{\varepsilon}_L - \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}_{NL}\right)dV \tag{1}$$

By relating the strains to the degrees of freedom $\mathbf{d}$ of the system, the total potential energy may be rewritten as given in Eq. (2), where $\mathbf{K}_{\mathrm{E}}$ is the global elastic stiffness matrix of the system and $\mathbf{K}_{\mathrm{G}}$ is the global geometric stability matrix of the system, which scales linearly with the applied stresses and so is often written as a load factor λ multiplied by $\mathbf{K}_{\mathrm{G}}$ calculated for some reference stresses. Obtaining the stiffness and stability matrices for the SAFSM is briefly described in the following sections.

$$U_e - W = \frac{1}{2}\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{E}}\mathbf{d} - \frac{1}{2}\mathbf{d}^{\mathrm{T}}\mathbf{K}_{\mathrm{G}}\mathbf{d} = \frac{1}{2}\mathbf{d}^{\mathrm{T}}(\mathbf{K}_{\mathrm{E}} - \lambda\mathbf{K}_{\mathrm{G}})\mathbf{d} \tag{2}$$

By making the total potential energy of Eq. (2) stationary with respect to each degree of freedom, the linear buckling eigenvalue problem of the SAFSM is formulated and is as given in Eq. (3), where $\boldsymbol{\Lambda}$ is a diagonal matrix containing the eigenvalues of the problem and $\boldsymbol{\Theta}$ is a matrix whose columns are the corresponding eigenvectors.

$$(\mathbf{K}_{\mathrm{E}} - \boldsymbol{\Lambda}\mathbf{K}_{\mathrm{G}})\boldsymbol{\Theta} = \mathbf{0} \tag{3}$$

### 2.2. Displacement fields

The first SAFSM able to analyse members under shear was that of Plank and Wittrick [3], who utilised complex degrees of freedom coupled with complex exponentials as defined by Eq. (4), where $\xi$ is proportional to the coordinate in the longitudinal direction and $i$ is the imaginary unit, to incorporate the phase-shift of displacements across the strip width that occurs for a member under shear stresses.

$$e^{i\xi} = \cos\xi + i\cdot\sin\xi \tag{4}$$

The displacement fields $d$ were then defined as,

$$d = \mathrm{Re}\{\mathbf{N}\mathbf{d}\cdot e^{i\xi}\} \tag{5}$$

where $\mathbf{N}$ is the vector of transverse shape functions for the current displacement field, $\mathbf{d}$ is the vector of corresponding complex degrees of freedom and 'Re' denotes the real part of its argument. When evaluating the longitudinal (warping) displacements, the argument of Eq. (5) was multiplied by $i$ to incorporate the out-of-phase nature of these displacements with respect to the transverse displacements. The utilised longitudinal functions correspond to a finite-length member with unrestrained ends or, equivalently, a member of infinite length with supported ends. Due to its complex mathematics, formulation of the stiffness and stability matrices depended on the identity given in Eq. (6), where $\mathbf{a}$ and $\mathbf{b}$ are vectors of equal length, $\mathbf{G}$ is a square matrix of corresponding size and the bar denotes the complex conjugate. This identity has no direct analog for the case where the arguments of the two complex exponentials differ and so coupling between longitudinal series terms is not possible.

$$\int_0^{2\pi} \mathrm{Re}\{\mathbf{a}^{\mathrm{T}}e^{i\xi}\}\cdot\mathbf{G}\cdot\mathrm{Re}\{\mathbf{b}e^{i\xi}\}\,d\xi = \pi\cdot\mathrm{Re}\{\bar{\mathbf{a}}^{\mathrm{T}}\mathbf{G}\mathbf{b}\} \tag{6}$$

One method to overcome this is to explicitly evaluate the 'realness' of Eq. (5) prior to formulating the stiffness and stability matrices. This was done by Mahendran and Murray [23] and resulted in a FSM with twice the usual number of degrees of freedom. Although this explicitly real displacement field of sines and cosines makes it possible to consider coupling between longitudinal series terms, Mahendran and Murray did not do so. The right-handed coordinate system and degrees of freedom utilised herein are as shown in Fig. 1; the corresponding displacement fields are as given in Eqs. (7)–(11), where $c_m$ and $s_m$ are as defined by Eq. (11) and so correspond to $m$ half-wavelengths along the considered length $L$, the superscripts '$r$' and '$i$' refer to the real and imaginary components of the complex degrees of freedom of Eq. (5), and $q$ is the number of series terms considered. Note that, due to the sign of the shape functions used for the rotational degrees of freedom, a positive rotation about the longitudinal axis is defined by the left-hand rule. The degrees of freedom in Fig. 1 are all shown at the mid-length of the strip,

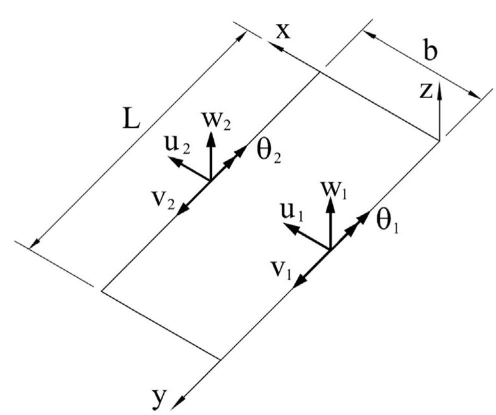


**Fig. 1.** Local degrees of freedom and axes of a finite strip.