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Reconciling Savage's and Luce's modeling of uncertainty: The best of both worlds

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HIGHLIGHTS

- Mosaics of events are more suited for modeling uncertainty than (σ-)algebras.
- We connect Luce's and Savage's ways of modeling uncertainty.
- Luce's modeling of uncertainty can be applied to modern decision theories.
- Most models of uncertainty can be embedded in Savage's model.

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ABSTRACT

This paper recommends using mosaics, rather than (σ-)algebras, as collections of events in decision under uncertainty. We show how mosaics solve the main problem of Savage's (1954) uncertainty model, a problem pointed out by Duncan Luce. Using mosaics, we can connect Luce's modeling of uncertainty with Savage's. Thus, the results and techniques developed by Luce and his co-authors become available to currently popular theories of decision making under uncertainty and ambiguity.

1. Introduction

Savage (1954) introduced the best-known and most-used model for decision under uncertainty, with gambles[1] mapping states to consequences. A decision maker chooses a gamble, nature independently chooses a state, and the corresponding consequence results. Duncan Luce pointed out some serious drawbacks to Savage's model. Throughout his career, Luce used the following example to illustrate these drawbacks. We use it as the lead example in our paper:

> If one is considering a trip from New York to Boston, there are a number of ways that one might go. Probably the primary ones that most of us would consider are, in alphabetical order, airplane,[2] bus, car, and train.

When you consider each transportation alternative, you can focus on the uncertainties relevant to that alternative only. However, Savage's model requires you to consider not only the separate uncertainties regarding each alternative, but also all joint uncertainties. Thus, when choosing between airplane and car, you have to consider your degree of belief that both the airplane and the car (had it been taken) would be delayed jointly. This joint event is, however, irrelevant to the decision to be made. Savage's requirement may lead to large and intractable event and gamble spaces. Further, the resolution of joint uncertainties often is not even observable. For instance, if you had chosen to travel by airplane, then you could never fully learn about the delays of the car trip, which did not even take place.

Luce developed various conditional decision models to avoid the aforementioned drawbacks. In the lead example, one then only considers the uncertainties relevant to (conditioned on) each

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1 This is Luce's term. Savage (1954) used the term act. We use Luce's (2000) terminology as much as possible.

2 The exact quote is from Luce (2000, Section 1.1.6.1). During his childhood, Luce was much interested in airplanes (besides painting), and he majored in aeronautical engineering. His parents advised against an art career, and astigmatism ruled out military flying, so that he turned to academic research. This history may have contributed to the adoption of this example. Luce used the example also in Krantz, Luce, Suppes, and Tversky (1971, Section 8.2.1) and Luce and Krantz (1971, Section 2).

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transportation alternative separately, with no need to inspect the irrelevant joint uncertainties. As pointed out by Luce and others, these models, while avoiding some problems, create other problems. As we will argue in further detail later, one drawback of Luce's models is that they do not have Savage's clean separation between chance (in nature) and the human will of the decision maker. In Luce's models, both the decision maker and nature may choose (conditioning) events to happen. Another drawback is that part of the mathematical elegance of Savage's model is lost (pointed out by Luce, 2000 p. 7 and discussed below).

This paper reconciles Savage's and Luce's models, with the aforementioned problems solved and the best of both worlds preserved. For this purpose, we propose a generalization of Savage's model, based on Kopylov's (2007)[3] mosaics. Mosaics relax the intersection-closedness requirement of algebras, which is the cause of the aforementioned problems in Savage's model. Using mosaics we can model Luce's lead example without considering irrelevant and inconceivable combinations of uncertainties. At the same time, we maintain Savage's mathematical elegance and his clear separation of nature's influence and the decision maker's influence. We will show that for every Luce (2000) model there exists an isomorphic Savage model, which implies that this isomorphic model can capture all structures and phenomena that Luce's model can, and it can do so in the same tractable manner. In addition, our model satisfies all principles of Savage's model: One state space captures all uncertainties, and the moves of nature and the decision maker are completely separated. In this sense, our model has the best of both worlds.

Our result shows the usefulness of mosaics. The main conclusion of this paper, entailing a blend of Savage's and Luce's ideas, extends beyond the reconciliation obtained. We recommend the use and study of mosaics rather than (σ-)algebras as the event spaces for decision under uncertainty in general. This raises a research question: To what extent can the appealing and useful mathematical results obtained for algebras in the literature be generalized to mosaics? Abdellaoui and Wakker (2005) and Kopylov (2007) provided several positive results.[4]

This paper is organized as follows. Section 2 discusses Savage's (1954) model and Section 3 discusses Luce's (2000) model, the most comprehensive account of his views. Our reconciliation of these two models is in Section 4. Section 5 overviews some other deviations from Savage's model, including Luce and Krantz (1971), which contributed to and preceded Luce (2000). Unlike Luce (2000), the models considered there are not isomorphic to (a generalization of) Savage's (1954) models, but we show that they can still be embedded (i.e., are isomorphic to substructures). Thus we show for all models considered how they can be related to the revealed preference paradigm of economics. Bradley (2007) provides a general logical model that can embed all models considered in this paper as substructures. Section 6 presents a discussion and Section 7 concludes. The Appendix discusses some other generalizations of Savage's model that Luce considered, being compounding, coalescing, and joint receipts, which are tangential to our main topic: connecting Luce's uncertainty model with other uncertainty models popular in the literature today. Our connection allows the introduction of Luce's techniques, including those in the Appendix and follow-up papers,[5] into modern decision theories.

2. Savage (1954)

This section reviews Savage's (1954) model. Savage models uncertainty through a state space S. One state $s \in S$ is true and the other states are not true, but it is uncertain which state is the true one. S is endowed with an algebra $\mathcal{E}$ of subsets called *events*.[6] An *algebra* contains S and is closed under union and complementation. It follows from elementary manipulations that an algebra also contains $\emptyset$ and is closed under finite unions and intersections. An event is *true* if it contains the true state of nature. $\mathcal{C}$ is a set of *consequences*; it can be finite or infinite. A decision maker has to choose between *gambles* (generic symbol G), which are mappings from S to $\mathcal{C}$ with finite image[7] that are measurable with respect to $\mathcal{E}$. *Measurability* of G means that for each consequence x its inverse under G, $G^{-1}(x)$, is an event. It implies that $G^{-1}(D)$ is an event for every subset $D \subset \mathcal{C}$: $G^{-1}(D)$ is a finite union of events $G^{-1}(x)$ of the elements $x \in D$, where only finitely many of these events $G^{-1}(x)$ are nonempty.

The decision maker's comparisons between gambles constitute a preference relation $\succcurlyeq$. Some approaches do not take states and consequences as primitives, with gambles derived, but take gambles and consequences, or gambles and states, as primitives (Fishburn, 1981 Section 8.4; Karni, 2006, 2013). Yet these approaches can be recast in terms of the original Savage model for the purposes of this paper (Schmeidler & Wakker, 1987).

Savage gave a preference foundation for expected utility theory:

$$G \to \int_S U(G(s))\, dP(s)\,. \tag{2.1}$$

Here $U : \mathcal{C} \to \mathbb{R}$ is a *utility function*, and P is a *probability measure* defined on the events. This paper does not discuss which particular decision theory (such as expected utility theory, prospect theory, multiple priors, and so on) is to be used. Its topic concerns the general modeling of uncertainty.

As regards Savage's drawback of involving a complicated event space, we not only have to specify all joint uncertainties but also have to posit axioms sufficiently wide-ranging to generate all likelihoods. Then, further, all gambles whose consequences are contingent on the complicated event space have to be considered. This drawback was elaborated by Luce (2000, p. 6):

> It is certainly not unreasonable to suppose that each mode of travel entails, as a bare minimum, at least 10 distinct [uncertain events].[8] To place this simple decision situation in the Savage

3 In 2007, Kopylov worked at the economics department of the University of California at Irvine, within a mile of Luce's office who was at the psychology department there. Yet Kopylov developed his idea independently of Luce's work.

4 Kopylov (2007), while working from a different motivation (see below), in fact already gave a positive answer for Savage's (1954) foundation of expected utility, by extending it to mosaics. Abdellaoui and Wakker (2005, written after and building on Kopylov's paper) provided further generalizations and preference foundations for a number of popular nonexpected utility theories for risk and uncertainty: the Gilboa (1987)–Schmeidler (1989)–Quiggin (1982) rank-dependent utility (including Choquet expected utility), Tversky & Kahneman's (1992) prospect theory (which applies not only to risk but also to ambiguity), and Machina & Schmeidler's (1992) probabilistic sophistication. Generalizations of other models of uncertainty to mosaics is a topic for future research, as is the extension of measure-theoretic concepts to mosaics.

5 References include (Liu (2003), Luce (2010), Luce and Marley (2005), Marley and Luce (2005), Marley, Luce, and Kocsis (2008)).

6 In his main analysis, Savage (1954) assumed that $\mathcal{E}$ is the power set, but he pointed out that it suffices that it is a σ-algebra (Section 3.4, pp. 42–43). His preference conditions, especially his P6, imply that S is infinite. Technical aspects such as the difference between σ-algebras and algebras are not important in this paper and we keep these aspects as simple as possible.

7 We throughout make this assumption, common in decision theory and made throughout Luce (2000, see his p. 3), to simplify the mathematics.

8 Luce instead used the term outcome. This term commonly refers to uncertain events (states of nature) in probability theory, a convention followed by Luce. In decision theory, however, the term outcome commonly refers to consequences rather than events. To avoid confusion, we do not use this term.