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Aggregation operators, comparison meaningfulness and social choice

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HIGHLIGHTS

- New insights about the links existing among aggregation theory, measurement theory, and social choice, are established.
- Two questions posed in the literature, concerning aggregation functions mapping independent ordinal scales into an ordinal scale, are answered.
- A multidimensional setup is considered and no continuity assumptions are required in any of the stated results.

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ABSTRACT

By exploring in more detail the links existing among aggregation theory, measurement theory, and social choice, some new results concerning aggregation operators are shown. This interplay was earlier observed and studied by Luce and later developed by many others. Two characterizations of aggregation operators, closely related to projections, are provided, thus generalizing the existing literature on these topics. It is emphasized that no continuity assumptions are required in any of the stated results.

1. Introduction

This paper aims to further explore the links existing among three well-established, and at first sight disparate, disciplines; namely, aggregation theory, measurement theory, and social choice. In particular, I present some results in aggregation theory that have largely taken their inspiration, in concepts and ideas, from measurement theory and social choice and that generalize the existing literature in certain respects. This interesting cross-fertilization interaction was noted by Luce in the late 1950s, and his pioneering and seminal contributions allowed for the development of a significant body of work in the years since then; including, specifically, the foundational paper by Luce (Luce, 1959) that establishes the connection between measurement theory and functional equations theory, and the book by Luce and Raiffa (Luce & Raiffa, 1957) in which the basis of utility theory in social choice is established. Further important contributions to these two research streams include (Aczél & Roberts, 1989; Aczél, Roberts, & Rosembaum, 1986; Kim, 1990; Marichal, 2002; Marichal & Mathonet, 2001; Osborne, 1970), in measurement theory, and (Bossert and Weymark, 2004; d'Aspremont & Gevers, 2002; Hammond, 1976; Krause, 1995; Roberts, 1980; Sen, 1977), in social choice theory.

The main concept employed throughout the paper is that of comparison meaningfulness with respect to independent ordinal scales. This turns out to be a natural concept in measurement theory when describing certain invariance properties, under a set of admissible transformations, of aggregation functions. This notion is extended here to a multidimensional setup. In the social choice literature, comparison meaningful statements are usually interpreted in terms of utility measurability and the possibility of making inter/intra-personal comparisons of well-being among individuals.

The motivation for the present work can be found in the two questions posed in Kim (1990) concerning one-dimensional aggregation functions that are comparison meaningful with respect to independent ordinal scales. On the one hand, the possibility of weakening the continuity assumption of an aggregation function representing a particular scientific or psycho-physical law, and, on the other hand, the interest in providing results under partial independence of the scales used to measure the variables. Both of these questions are answered in the current paper.

Here is a brief outline of the contents of the article. Section 2 includes the basics of the aggregation framework that will be developed. Let there be given a nonempty set X, and a natural number $n \geq 1$. Then an *aggregation operator* is a map $T : \mathcal{U}^n \to \mathcal{U}$, where $\mathcal{U}$ denotes the set of all real-valued functions defined on X. When X is finite, with at least two points, and $n > 1$, this definition agrees with that of a *social evaluation functional* introduced and studied in Candeal (2015) within a social choice

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framework.[1] Another interesting application of this formalism is provided by the grading functions used in Balinski and Laraki (2007) in the context of majority judgment.

Section 3 is devoted to the study of two particular classes of aggregation operators; namely, the class of representable and the class of weakly representable aggregation operators, respectively. The corresponding characterizations of these two kinds of operators are established by adapting certain axioms, that appear in the social choice literature, to this setup.

Section 4 presents the main results of the paper. It focuses on aggregation operators that are comparison meaningful with respect to independent ordinal scales. Two types of results are then presented. In one, the implications of adding some separability property to these aggregation operators is considered. In the other, and regardless of any separability property, I study aggregation operators that satisfy some monotonicity axiom; namely, those that are nondecreasing. Both statements lead to a similar conclusion providing characterizations of aggregation operators which are closely related to projections. The results obtained allow me to give precise answers to the questions posed in Kim (1990). In particular, it is proven that the continuity assumption given in Theorem 1 of Kim (1990) can be dispensed with. It is worth mentioning that the proofs are mainly based upon order-theoretical principles, and no continuity assumption is used. Although certain ideas of the article come from utility theory and social choice theory, no particular application to any of these fields is developed. That is left for a future research.

2. Aggregation operators: first definitions

In this section the primitives involving the concept of an aggregation operator, that will be considered throughout the paper, are introduced. I closely follow the notations used in Candeal (2015).

Let n, m be two natural numbers. Then $N := \{1, \ldots, n\}$, and $M := \{1, \ldots, m\}$. As usual $\mathbb{R}^n$ will denote the n-dimensional Euclidean space, i.e., $\mathbb{R}^n = \{a = (a_j) : a_j \in \mathbb{R},\ j \in N\}$. The usual binary operations in $\mathbb{R}^n$ of addition, multiplication by scalars, and vector multiplication, defined coordinatewise, will be denoted by $+, \cdot_{\mathbb{R}}$ and $*$, respectively. Given $a = (a_j), b = (b_j) \in \mathbb{R}^n$, I write $a \leq b$ whenever $a_j \leq b_j$ for all $j \in N$, $a \ll b$ whenever $a_j < b_j$ for all $j \in N$ and $a < b$ whenever $a \leq b$ and, for some $j \in N$, $a_j < b_j$. The null vector of $\mathbb{R}^n$ will be denoted by 0_n.

Let X be a nonempty set. A typical function, from X to $\mathbb{R}$, will be denoted by u. The set of all real-valued functions from X to $\mathbb{R}$, usually denoted by $\mathbb{R}^X$, will be here denoted by $\mathcal{U}$.

A *n-tuple* (also called a *profile*) of real-valued functions will be denoted by $U = (u_j)_{j \in N}$ (or simply, by $U = (u_j)$), and $\mathcal{U}^n$ will stand for the set of all possible n-tuples. A profile $U = (u_j) \in \mathcal{U}^n$ can also be viewed as a real-valued map defined on $X \times N$ in the following manner: $(x, j) \in X \times N \longrightarrow U(x, j) = u_j(x) \in \mathbb{R}$. So, in the case that $X = M$ a profile $U = (u_j)$ can be identified with an $m \times n$ matrix whose entries are the real numbers $(u_j(i))_{i \in M,\ j \in N}$.[2]

In order to present some basic definitions the following notation will be useful. For a given $x \in X$, and $U = (u_j) \in \mathcal{U}^n$, $U(x)$ will denote the following vector in $\mathbb{R}^n$, $U(x) := (u_j(x))$. In addition, and as above, for each $j \in N$, $U(x, j)$ stands for the real number $U(x, j) := u_j(x)$. Let $\Omega := \{\phi : \mathbb{R} \to \mathbb{R}\}$ and let $a = (a_j) \in \mathbb{R}^n$. By $\phi(a)$ I mean the following vector of $\mathbb{R}^n$, $\phi(a) := (\phi(a_j)), j \in N$.

In a similar way, let $\Phi = (\phi_j) \in \Omega^n$ denote an n-tuple of real-valued functions defined on $\mathbb{R}$ (i.e., $\phi_j \in \Omega$, for every $j \in N$). Then $\Phi(a) := (\phi_j(a_j)) \in \mathbb{R}^n$. If $\Phi = (\phi_j) \in \Omega^n$ and $U = (u_j) \in \mathcal{U}^n$, then $\Phi \circ U := ((\phi u)_j) = (\phi_j \circ u_j) \in \mathcal{U}^n$, where "$\circ$" stands for the usual composition operation, i.e., $\phi_j \circ u_j(x) = \phi_j(u_j(x))$, for all $x \in X, j \in N$. For ease of notation, at times I will use ΦU instead of $\Phi \circ U$. Note that, with the notation introduced, it holds that $\Phi U(x) = \Phi(U(x))$, for every $x \in X$. If the n-tuple $\Phi = (\phi_j) \in \Omega^n$ has the property that $\phi_k = \phi_l = \phi$, for all $k, l \in N$, for a given $\phi \in \Omega$, then I will write $\Phi = \phi 1_n$. Similarly, for a given $u \in \mathcal{U}$, $U_u := u 1_n = (u, \ldots, u) \in \mathcal{U}^n$.

The sub-domain of Ω which consists of all strictly increasing real-valued functions will be denoted by Δ. The n-fold Cartesian product of this space will be denoted by Δ^n. An important sub-domain of Δ, that will be considered in the sequel, consists of all strictly increasing affine real-valued functions, denoted by Δ_{ia}, i.e., $\Delta_{ia} = \{\phi \in \Delta : \phi(t) = at + b, a > 0, b \in \mathbb{R}\}$.

An *n-dimensional aggregation operator*, or simply an aggregation operator, is a map $T : \mathcal{U}^n \to \mathcal{U}$. An aggregation operator T is said to be *constant* if there is $u_0 \in \mathcal{U}$ such that $T(U) = u_0$, for all $U \in \mathcal{U}^n$, (i.e., T is a constant map). In the case that u_0 is also a constant function, then T is said to be *strongly constant*.

Let T be an aggregation operator and let $x \in X$. Then the pair (T, x) induces a real-valued function defined on $\mathcal{U}^n$, denoted by T^x, in the following way: $U \in \mathcal{U}^n \longrightarrow T^x(U) = T(U)(x) \in \mathbb{R}$. Each of these functions T^x will be called a *component function* of T. Note that, if X is finite, say $X = M := \{1, \ldots, m\}$, then T can be viewed as a vector-valued function defined on the space of all $m \times n$ real matrices and, as codomain, the space of all real column matrices of size $m \times 1$. In this case, each of the component functions of T, say $T^i, i \in M$, is a real-valued map of $m \times n$ real-variables.

Definition 2.1. An aggregation operator $T : \mathcal{U}^n \to \mathcal{U}$ is said to be:

(i) *idempotent* if $T(U_u) = u$, for all $u \in \mathcal{U}$,
(ii) *weakly idempotent* if, for each $r \in \mathbb{R}$, $T(U_{u^r}) = u^r$, where $u^r(x) = r$, for all $x \in X$,
(iii) *projective* if it is a projection, i.e., if there is $j \in N$ such that $T(U)(x) = U(x, j)$, for all $U \in \mathcal{U}^n, x \in X$,
(iv) *weakly projective* if for each $x \in X$, T^x is a projection; i.e., for each $x \in X$ there is $j_x \in N$ such that $T^x(U) = T(U)(x) = U(x, j_x)$, for all $U \in \mathcal{U}^n$.

Remark 2.2. For an aggregation operator $T : \mathcal{U}^n \to \mathcal{U}$, the following chain of implications holds: projective $\Rightarrow$ weakly projective $\Rightarrow$ idempotent $\Rightarrow$ weakly idempotent.

3. Representable and weakly representable aggregation operators

In this section two important kinds of aggregation operators are introduced; namely, the class of representable aggregation operators and the more general class of weakly representable aggregation operators. As will be seen, these two types of aggregation operators convey interesting properties when considering the aggregation problem in the social choice theory context. I begin with a basic and key definition.

Definition 3.1. Let $T : \mathcal{U}^n \to \mathcal{U}$ be an aggregation operator. Then T is said to be:

(i) *representable* if there is a real-valued function $R : \mathbb{R}^n \to \mathbb{R}$ such that $T(U)(x) = R(U(x))$, for every $U \in \mathcal{U}^n, x \in X$,
(ii) *weakly representable* if there exists a real-valued function $W : X \times \mathbb{R}^n \to \mathbb{R}$ such that $T(U)(x) = W(x, U(x))$, for every $U \in \mathcal{U}^n, x \in X$.

[1] In the context of social choice theory, the set X is usually called the set of alternatives, or the set of social states, and n is the number of agents or individuals in the society.

[2] In the context of social choice theory, for each profile $U \in \mathcal{U}^n$, $U(i, j)$ represents the value that individual j assigns to alternative i.