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Explicit moments of decision times for single- and double-threshold drift-diffusion processes

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HIGHLIGHTS

- Analytic expressions for first three unconditioned and conditioned moments of decision time for pure drift-diffusion model.
- Semi-analytic expressions for first three unconditioned and conditioned moments of decision time for extended drift-diffusion model.
- Thorough analysis of the behavior of moments of decision time as a function of model parameters.
- Analysis of the effect of non-decision time on moments of reaction time.

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ABSTRACT

We derive expressions for the first three moments of the decision time (DT) distribution produced via first threshold crossings by sample paths of a drift-diffusion equation. The "pure" and "extended" diffusion processes are widely used to model two-alternative forced choice decisions, and, while simple formulae for accuracy, mean DT and coefficient of variation are readily available, third and higher moments and conditioned moments are not generally available. We provide explicit formulae for these, describe their behaviors as drift rates and starting points approach interesting limits, and, with the support of numerical simulations, discuss how trial-to-trial variability of drift rates, starting points, and non-decision times affect these behaviors in the extended diffusion model. Both unconditioned moments and those conditioned on correct and erroneous responses are treated. We argue that the results will assist in exploring mechanisms of evidence accumulation and in fitting parameters to experimental data.

1. Introduction

In this paper we derive explicit expressions for the mean, variance, coefficient of variation and skewness of decision times (DTs) predicted by the stochastic differential equation (SDE)

$$dx = a\,dt + \sigma\,dW, \quad x(0) = x_0, \tag{1}$$

which models accumulation of the difference $x(t)$ between the streams of evidence in two-alternative forced-choice tasks. An example of such a perceptual decision-making task is one in which a participant determines if the image on the screen has more white or black pixels (e.g., Ratcliff & Rouder, 1998). Here drift rate a and standard deviation σ are constants, dW denotes independent random (Wiener) increments, and dx is the change in evidence during the time interval $(t, t + dt)$. Decision times (DTs) are determined by first passages through upper and lower thresholds $x = +z$ and $-z$ that respectively correspond to correct responses and errors, between which the starting point x_0 is assumed to lie. Thus, without loss of generality we may set $a > 0$, although we will also consider limits $a \to 0$. Predictions of response times (RTs) for comparison to behavioral data are obtained by adding to DTs a non-decision latency, T_{nd}, to account for sensory and motor processes.

SDEs like Eq. (1) are variously called diffusion or drift-diffusion models (DDMs); in Bogacz, Brown, Moehlis, Holmes, and Cohen (2006) Eq. (1) was named the pure DDM to distinguish it from Ratcliff's extended diffusion model (Ratcliff, 1978), which allows trial to trial variability in drift rates and starting points x_0. See Bogacz et al. (2006), Ratcliff (1978) and Ratcliff and Smith (2004) for background on diffusion models, and note that several different variable-naming conventions are used in parameterizing DDMs, e.g. in Ratcliff (1978), Ratcliff and Smith (2004) and Wagenmakers,

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Grasman, and Molenaar (2005) v and s replace a and σ, and thresholds are set at $x = 0$ and $x = a$ with $x_0 \in [0, a]$; in Bogacz et al. (2006) a and σ are named A and c.

Many of the following results have appeared in the stochastic processes literature, or are implicit in it, and some have appeared in the psychological literature (e.g. Grasman, Wagenmakers, & van der Maas, 2009; Ratcliff, 1978; Wagenmakers et al., 2005). However, their dependence on key parameters such as threshold and starting point and behaviors in the limits of low and high drift rates have not been fully explored (see Wagenmakers et al., 2005 for some cases of $a \to 0$). Nor are we aware of explicit derivations of third order moments. Here we provide these, and also prove a Proposition that describes the structure of the coefficient of variation (CV) for DTs predicted by Eq. (1), relating it to the CV for a single-threshold DDM. We end by considering the extended DDM, introduced in Ratcliff (1978), showing how trial-to-trial variability of drift rates and starting points affects the results for the pure DDM and examining the effects of non-decision latency on response times. We summarize the expressions for the unconditioned and conditioned moments of DTs for the pure DDM in Table 1. The MatLab and R implementation of analytic and semi-analytic expressions for the conditioned and unconditioned moments of DTs for the pure and extended DDMs studied here is available at: https://github.com/PrincetonUniversity/higher_moments_ddm.

Notation and units

We start by reviewing definitions and dimensional units, and establishing notation. For a random variable ξ, we define the nth non-central moment by $\mathbb{E}[\xi^n]$ and the nth central moment by $\mathbb{E}[(\xi - \mathbb{E}[\xi])^n]$. The first central moment is zero and the second central moment is the variance. The coefficient of variation (CV) of ξ is defined as the ratio of standard deviation to mean of ξ, i.e., $\mathrm{CV} = \sqrt{\mathbb{E}[(\xi - \mathbb{E}[\xi])^2]}/\mathbb{E}[\xi]$. Similarly, the skewness of ξ is defined as the ratio of the third central moment to the cube of the standard deviation of ξ:

$$\text{skew} = \frac{\mathbb{E}[(\xi - \mathbb{E}[\xi])^3]}{\mathbb{E}[(\xi - \mathbb{E}[\xi])^2]^{3/2}}.$$

The variable $x(t)$ and thresholds $\pm z$ in Eq. (1) are dimensionless, while the parameters a and σ have dimensions $[\text{time}]^{-1}$ and $[\text{time}]^{-\frac{1}{2}}$ respectively. When providing numerical examples we will work in secs. For $a > 0$ we define the normalized threshold k_z and starting point k_x:

$$k_z = \frac{az}{\sigma^2} \geq 0 \quad \text{and} \quad k_x = \frac{ax_0}{\sigma^2} \in (-k_z, k_z); \tag{2}$$

these nondimensional parameters will allow us to give relatively compact expressions.

2. The single-threshold DDM

Eq. (1) with a single upper threshold $z > 0$ necessarily produces only correct responses in decision tasks, but it is of interest because it provides a simple approximation of the double-threshold DDM when accuracy is at ceiling and errors due to passages through the lower threshold are rare. Specifically, for $a > 0$, DTs of this model with starting point x_0 are described by the Wald (inverse-Gaussian) distribution (Borodin & Salminen, 2002, Eq. (2.0.2); Luce, 1986; Wald, 1947):

$$p(t) = \frac{z - x_0}{\sigma} \sqrt{\frac{1}{2\pi t^3}} \exp\left(\frac{-(z - x_0 - at)^2}{2\sigma^2 t}\right). \tag{3}$$

The mean DT, its variance, and CV are:

$$\mathbb{E}[\mathrm{DT}] = \frac{\sigma^2}{a^2}(k_z - k_x), \qquad \mathrm{Var}[\mathrm{DT}] = \frac{\sigma^4}{a^4}(k_z - k_x), \quad \text{and}$$
$$\mathrm{CV} = \frac{\sqrt{\mathrm{Var}[\mathrm{DT}]}}{\mathbb{E}[\mathrm{DT}]} = \frac{1}{\sqrt{k_z - k_x}}, \tag{4}$$

and the skewness is

$$\frac{3}{\sqrt{k_z - k_x}} \quad (= 3\,\mathrm{CV}). \tag{5}$$

In the limit $a \to 0^+$, the distribution (3) converges to the Lévy distribution, and in this limit none of the moments exist. However, as shown below, moments of the double threshold DDM exist in this limit.

The single threshold process has been proposed as a model for interval timing (Balci & Simen, 2014; Luzardo, Ludvig, & Rivest, 2013; Simen, Balci, deSouza, Cohen, & Holmes, 2011; Simen, Vlasov, & Papadakis, 2016). Interval timing, loosely defined, is the capacity either to make a response or judgment at a specific time relative to some event in the environment, or simply to estimate inter-event durations. Classic timing tasks include "production" tasks, such as the Fixed Interval (FI) task, in which a participant receives a reward for any response produced after a delay of a given duration since the last reward was received (Ferster & Skinner, 1957), and discrimination tasks, in which two different stimulus durations are compared to see which is longer (see Creelman, 1962 and Treisman, 1963 for historical reviews of early human timing research). Production tasks can be modeled similarly to decision tasks by a diffusion model: instead of accumulating evidence about a perceptual choice, a timing diffusion model accumulates a steady "clock signal" toward a threshold for responding (Creelman, 1962; Gibbon, Church, & Meck, 1984; Killeen & Fetterman, 1988; Treisman, 1963). The resulting production times, relative to stimulus onset, are then comparable to perceptual decision-making response times, typically yielding a slightly positively skewed Gaussian density (Gibbon & Church, 1990). Simen, Rivest, Ludvig, Balci, and Killeen (2013) show that the single-threshold DDM can fit RT data from a variety of interval timing experiments when the starting point is set to 0, drift is set equal to threshold over duration ($a = z/T$, with T = target duration), and normalized thresholds k_z are set to high values, typically of order 20 (see Simen et al., 2011). In contrast, k_z is usually much lower in fits of typical two-choice decision data, typically of order 1. Noise σ is typically fixed at 0.1 in the literature (Vandekerckhove & Tuerlinckx, 2007) and fitted thresholds typically range from 0.05 to 0.15; see, e.g. Balci et al. (2011), Bogacz, Hu, Holmes, and Cohen (2010), Dutilh, Vandekerckhove, Tuerlinckx, and Wagenmakers (2009) and Ratcliff (2014). Despite this difference, DDM can be fitted to both two-choice decision RTs and timed production RTs in humans with suitably larger thresholds for timing (Simen et al., 2016), suggesting that both tasks may be accomplished by similar accumulation processes.

3. The double-threshold DDM: Unconditioned moments of decision time

We now turn to the double-threshold DDM and derive unconditioned moments of decision time. The DT distribution for the double-threshold DDM may be expressed as a convergent series (Ratcliff, 1978, Appendix), and successive moments of the unconditioned DT (i.e. averaged over correct responses and errors) may be obtained by solving boundary value problems for a sequence of linear ordinary differential equations (ODEs) derived from the backwards Fokker–Planck or Kolmogorov equation (Gardiner, 2009, Chap. 5).