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Primary facets of order polytopes

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ABSTRACT

Mixture models on order relations play a central role in recent investigations of transitivity in binary choice data. In such a model, the vectors of choice probabilities are the convex combinations of the characteristic vectors of all order relations of a chosen type. The five prominent types of order relations are linear orders, weak orders, semiorders, interval orders and partial orders. For each of them, the problem of finding a complete, workable characterization of the vectors of probabilities is crucial—but it is reputably inaccessible. Under a geometric reformulation, the problem asks for a linear description of a convex polytope whose vertices are known. As for any convex polytope, a shortest linear description comprises one linear inequality per facet. Getting all of the facet-defining inequalities of any of the five order polytopes seems presently out of reach. Here we search for the facet-defining inequalities which we call primary because their coefficients take only the values −1, 0 or 1. We provide a classification of all primary, facet-defining inequalities of three of the five order polytopes. Moreover, we elaborate on the intricacy of the primary facet-defining inequalities of the linear order and the weak order polytopes.

1. Introduction

As it is the case in general for testing a deterministic theory on random sample data (Luce, 1959, 1995, 1997), checking whether transitivity is confirmed by a collection of two-item comparisons raises several interesting issues. Binary choice data usually consist of the relative frequencies of choice among any two alternatives. A formal approach to test whether the relative frequencies are consistent with transitivity relies on probabilistic models and derived statistical tests. Marley and Regenwetter (2016) and Regenwetter and Davis-Stober (2008) survey models for binary choice (forced or non-forced). We focus here on the characterization problem of the choice probabilities predicted by five of the main models.

A random utility model of binary choice relates the probability of choosing alternative i over j to the probability that the utility of i, taken as a random variable, exceeds that of j. As known since a long time for the direct comparison of random utility values,[1] the model happens to be a mixture model on linear orders (Block & Marschak, 1960). In precise terms, vectors of binary choice probabilities coincide with convex combinations of the characteristic vectors of linear orders on the set of alternatives. Recent work (Regenwetter & Davis-Stober, 2012b,a; Regenwetter & Marley, 2001), extends the traditional setting of linear orders to various types of order relations, principally weak orders, semiorders, interval orders and partial orders (the meaning of the terms will be explained in the next section). The random utility model, based on a specific, modified way of comparing two random utility values, admits a reformulation as a mixture model of order relations.

One of the fundamental problems on probabilistic models is to find out a workable characterization of the (probabilistic) predictions it makes. In the case of the mixture models of order relations, the characterization plays an important role in implementing tests of transitivity on binary choice data. However, complete characterizations were obtained only when the number of alternatives is small (we give details in Section 3). Even for the particular case of linear orders (in a way the most structured of our relations), the characterization problem raises mathematical difficulties and is widely seen as intractable. Section 3 recalls some explanations for the latter assertion, and moreover indicates why similar difficulties appear for the other four types of order relations.

For mixture models, as the ones we investigate here, a geometric point of view is most useful. Indeed, a characterization of the model is akin to the description of a certain polytope. More precisely, the polytope is given by its vertices (in our cases, the characteristic vectors of the order relations) and the aim

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[1] Under the assumption that equality of the utilities of two distinct alternatives occur with probability zero.

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is to describe the polytope as the solution set of a system of linear (in)equalities. If such a linear description is moreover a shortest one, then the number of linear equations is equal to the codimension of the polytope, and there is one linear inequality per facet of the polytope. This shows the importance of facet-defining inequalities, or FDI's. We refer the reader to Section 2 for a short summary of the concepts and results we will need and to Ziegler (1998) for more background on general convex polytopes. The five types of order relations we mentioned before lead to five convex polytopes: the linear ordering polytope, the weak order polytope, the interval order polytope, the semiorder polytope and the partial order polytope. In the past, the first polytope received the most attention (we provide references in the sections dedicated to the respective polytopes but we want to mention here an unpublished manuscript of Suck, 1995, the first to promote a common approach to order polytopes). It is recognized that obtaining a full, linear description of any of these five polytopes is a very difficult problem (cf. Section 3). We found it interesting to investigate the facet-defining inequalities with coefficients in the set $\{-1, 0, 1\}$, with the aim of assessing the relative difficulties in the five cases. A linear inequality is *primary* when its coefficients (including the independent term) take only the values -1, 0 or 1. Here is a summary of our results.

As a warm-up exercise, we provide a complete description of the primary linear inequalities which define facets of the partial order polytope (Section 5) or the interval order polytope (Section 6). Then we present a rather satisfiable understanding of the FDI's of the semiorder polytope; here, the results turn out to be rather technical (see Sections 7–9). We see the cases of the strict weak order and the linear ordering polytopes to be out of our reach even for primary linear inequalities, as we explain in Sections 10 and 11.

Two directions of possible further research are worth mentioning here. First, techniques from combinatorial optimization could be applied to the primary linear inequalities found here to derive more inequalities (such as those resulting from so-called Chvátal–Gomory cuts; for an introduction to the techniques, see Bertsimas & Weismantel, 2005, Section 9.4, or Conforti, Cornuéjols, & Zambelli, 2014, Chapter 5). Second, when a polytope Q contains a polytope P, new FDI's of one of the two polytopes can sometime be inferred from FDI's of the other polytope; we leave for future work the related inspection of the inclusions among our five order polytopes.

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2. Background: types of order relations and their polytopes

In this section we briefly recall some basic facts, first about order relations, then about polytopes. Throughout the paper, n denotes a natural number with $n \geqslant 2$. We write $[n]$ for the set $\{1, 2, \ldots, n\}$ of *elements* (or alternatives). Moreover, we denote by A_n the set of pairs of distinct elements, that is:

$$A_n = \{(i, j) \mid i, j \in [n], i \neq j\}.$$

Let R be an irreflexive binary relation on $[n]$ (that is, $R \subseteq A_n$); we write iRj for $(i, j) \in R$. Then R is a *(strict) partial order* if R is asymmetric (that is, if iRj then not jRi) and transitive (if iRj and jRk then iRk). A *linear order* is a partial order which is total (two distinct elements are always comparable). A *strict weak order* is a partial order which is *negatively transitive* (that is, iRk implies iRj or jRk). (Notice that strict weak orders are the complements of 'complete preorders', as we explain in Section 10.) An *interval order* S on $[n]$ is a partial order for which there exist two maps f and g from $[n]$

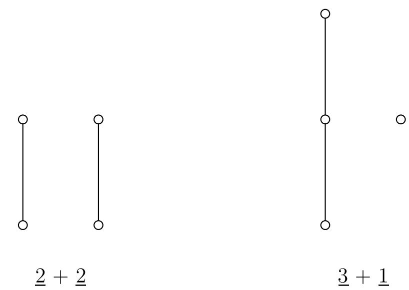

Fig. 1. The Hasse diagrams of the posets $\underline{2} + \underline{2}$ and $\underline{3} + \underline{1}$.

to $\mathbb{R}$ such that

iSj if and only if $g(i) < f(j)$

(thus iSj exactly if the closed interval $[f(i), g(i)]$ of the real line is located entirely below the similar interval $[f(j), g(j)]$). The pair (f, g) of maps is then called an *interval representation* of S. If S admits an interval representation (f, g) such that $g(i) = f(i) + 1$ for each i in $[n]$ (every interval has length 1) then S is also called a *semiorder*; we then say that f is a *unit interval representation*. We now state two classical theorems characterizing interval orders and semiorders (for the proofs as well as additional basic terminology, see a textbook as for example Fishburn, 1985 or Trotter, 1992). It is easy to check that the partial orders represented by their Hasse diagrams in Fig. 1 are not semiorders; we denote them by $\underline{2} + \underline{2}$ and $\underline{3} + \underline{1}$ respectively. The second one is an interval order, while the first one is not.

Theorem 1 (*Fishburn Theorem*)**.** *A partial order is an interval order if and only if it does not induce any* $\underline{2} + \underline{2}$.

Theorem 2 (*Scott–Suppes Theorem*)**.** *A partial order is a semiorder if and only if it does not induce any* $\underline{2} + \underline{2}$ *nor* $\underline{3} + \underline{1}$.

Obviously, any strict weak order is a semiorder, any semiorder is an interval order, and any interval order is a partial order.

We now move on to convex polytopes. A detailed treatment of the subject can be found for example in Ziegler (1998). A *convex polytope* in some space $\mathbb{R}^d$ is the convex hull of a finite set of points. For X a finite subset of $\mathbb{R}^d$, let P be the polytope

$$P = \operatorname{conv}(\{x_v \mid v \in V\}).$$

The *dimension* $\dim(P)$ of P is the dimension of its affine hull (notice that, except otherwise mentioned, dim designates the affine dimension). A linear inequality on $\mathbb{R}^d$,

$$\sum_{i=1}^{d} \alpha_i x_i \leqslant \beta,$$

is *valid* for P if it is satisfied by all points of P. A *face* of P is the subset of points of P satisfying a given valid inequality with equality (thus $\varnothing$ and P are faces of P); then the inequality *defines* the face. The faces of P are themselves polytopes. The *vertices* of P are the points v such that $\{v\}$ is a face of dimension 0; the *facets* are the faces of dimension $\dim(P) - 1$. A valid inequality is *facet-defining* (or a *FDI*) if it defines a facet of P. The importance of the latter concept is clear from the following result. Assume that the polytope P is *full*, that is, of dimension d. Then P equals the set of solutions to all of its facet-defining inequalities; moreover, any system of linear inequalities on $\mathbb{R}^d$ whose set of solutions equals P necessarily contains all of the facet-defining inequalities.

The five polytopes we consider in the paper are defined in the space $\mathbb{R}^{A_n}$, where as before $A_n = \{(i, j) \mid i, j \in [n], i \neq j\}$. The next lemma recalls, in this setting, an easy result on valid inequalities. Notice how, for a vector $x \in \mathbb{R}^{A_n}$ and $(i, j) \in A_n$, we abbreviate $x_{(i,j)}$ into x_{ij}.

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