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Coherence conditions for preference modeling with ordered points

Meltem Öztürk *

Université Paris Dauphine, PSL Research University, CNRS, Lamsade, 75016, Paris, France

HIGHLIGHTS

- Introduction of coherence conditions in the general framework of preference modeling with intervals.
- Generalization of coherence conditions in the case of intervals with n points.
- Two special types of coherence conditions are analyzed: monotony and regularity.
- Some results on their numerical representation and related preference structures are obtained.
- Analysis of coherence conditions for intervals with 2, 3 or 4 points is done.

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ABSTRACT

In this article results on comparison rules on n ordered points are extended by introducing some coherence conditions. The main results concern a general characterization of coherence conditions and their influence on the 2-point, 3-point and 4-point interval representations of preference structures.

1. Introduction

This article concerns preference modeling using n ordered points as numerical representation. This topic was introduced by Tsoukiàs and Vincke (2004) and revisited by Öztürk and her colleagues (Öztürk, Pirlot, & Tsoukiàs, 2011) who developed a general framework. In this study the above general framework is extended by analyzing the use of different types of coherence conditions.

A numerical representation of a preference structure with n ordered points could be useful in applications with ordered data as illustrated later; it could also propose flexible and sophisticated models to decision-makers to illustrate their preferences.

To introduce preferences in a mathematical model, a formal, generally numerical, representation is needed. Numbers are ordered by their nature and facilitate the choice of the best alternative. In addition, the numbers are very commonly used by everybody, their interpretation is thus simple and they are perceived as a sign of objectivity or rationality. However, experts confronted with a real world issue, know that the preferences do not always correspond to a total or weak order. Of course, this does not imply that everyone is irrational nor that their preferences are haphazard.[1] Many researchers try to explain this phenomenon by analyzing the transitivity of relations and their completeness (see for instance Anand, 1987; Bordley & Hazen, 1991; Deparis, Mousseau, Öztürk, Pallier, & Huron, 2012; Fishburn, 1991; Kacprzyk & Roubens, 1988; Luce, 2000; Myung, Karabatsos, & Iverson, 2005; Tversky, 1969). Different models and axiomatizations have been proposed. Lexicographic semiorder (Tversky, 1969), Skew-symmetric bilinear utility (Fishburn, 1982), the priority heuristic model (Brandstaetter, Gigerenzer, & Hertwig, 2006) are a few of the intransitive models. Another interpretation of the violation of transitivity can be attributed to the fact that total and weak orders remain too strong to represent preferences. Thus, more flexible preference structures can be found in literature. The better known ones are partial orders, interval orders and semiorders (Fishburn, 1973; Luce, 1956; Pirlot & Vincke, 1997; Trotter, 1992). Other structures, such as biorders, split interval orders, tolerance orders, have also been proposed (see for instance Bogart, Jacobson, Langley, & F. R. McMorris, 2001; Doignon, Monjardet, Roubens, & Vincke., 1986; Fishburn & Trotter, 1999; Fishburn, 1997). The general framework presented in (Öztürk

[1] For instance, Kahneman said that *"choices are not nearly as coherent as the notion of a preference order would suggest, but they are also far from random"* (Preface, page xvi, (Kahneman & Tversky, 2000))

* Fax: + 33 1 44 05 40 91.
E-mail address: ozturk@lamsade.dauphine.fr.

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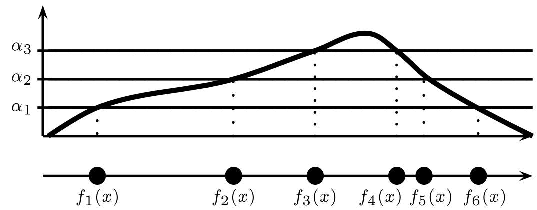

Fig. 1. Fuzzy number x, three of its α-cuts ($\alpha_1, \alpha_2, \alpha_3$) and its associated 6-point interval.

et al., 2011) concerns such structures and provides them with a common language.

Formally, n points $(f_1(x), \ldots, f_n(x))$ such as $\forall i, f_i(x) \leq f_{i+1}(x)$, are associated to each object x of A. Such a representation can also be interpreted as an interval with $n-2$ interior points. Therefore, these objects are called *n-point intervals* and $I_n(x)$ denotes the n-point interval for object x. The set of these intervals represents an *n-point interval representation* of A ($I_n(A) = U_{x \in A} I_n(x)$). Unless otherwise stipulated, the same notation, typically x or y, is used to designate an alternative or its associated interval. Two n-point interval representations are *equivalent* if the points of one can be obtained by a strictly increasing transformation of the points of the other, hence the ranking of all the points remains the same.

In the following, several contexts in which a representation of an object by n ordered points appears natural will be presented:[2]

i. When the evaluation of an object has many possibilities such as the price of an object in n shops.
ii. When the evaluation of an object is a fuzzy number (see Dubois and Prade (1983) for a ranking with possibility theory). In such a case, using $\frac{n}{2}$ α-cuts, n ordered points can be obtained (see for an example Fig. 1).
iii. When some special points are used as thresholds in order to define preference intensities. For instance, in case of *PQI* interval orders Tsoukiàs and Vincke, 2003, three points are used: a is strictly preferred to b iff $f_1(a) > f_3(b)$ and a is weakly preferred to b iff $f_1(a) > f_2(b)$, remaining situations are identified as indifference.
iv. In the context of decision under complete uncertainty where there is no knowledge about the probabilities (see for instance Barbera, Bossert, and Pattanaik, 2004) and where the correspondence between scenarios and consequences is not defined or not considered as in the case of the use of OWA aggregating operator which orders consequences with no reference to their relation with the scenarios.
v. When a lottery has a uniform probability distribution with n possibilities (for a more general view see Kothiyal, Spinu, and Wakker, 2014).

However, the analysis of different types of coherence conditions has not been considered in Öztürk et al. (2011). Coherence conditions impose some additional hypothesis on the position of ordered points, for instance imposing that all points are equidistant. In fact, in literature some preference structures, such as semiorders, split semiorders, unit tolerance orders, are defined using coherence conditions. Such conditions guarantee certain mathematical properties[3] and are relevant in industrial applications (like the famous example of sugar in the cup of coffee of Luce Luce, 1956). Thus, the study of Öztürk et al. (2011) is incomplete and in this paper I wish to fill this void.

In the following I reconsider the above examples and add some coherence conditions:

- For instance, in context (i.), if a 3-point interval represents the price of shampoos in a cheap, an average and an expensive shop, it can be reasonable to assume that if $f_1(x) < f_1(y)$ then $f_2(x) < f_2(y)$ and $f_3(x) < f_3(y)$.
- In context (ii.), if the data is in the form of an isosceles trapezoid fuzzy number then 4-point intervals corresponding to α-cuts ($\alpha = 1$ and $\alpha' = 0$) of the fuzzy number will satisfy $f_2(x) - f_1(x) = f_4(x) - f_3(x), \forall x$.
- In context (iii.), within *PQI* interval orders, the difference $f_2 - f_1$ (resp. $f_3 - f_1$) represents a threshold for weak preference (resp. strict preference). In such a case, decision-makers can impose constant values. For instance, when the prices of two cars are compared, if the difference between both is less than 500 euros, the decision-maker is indifferent, if this difference is between 500 euros and 1000 euros he weakly prefers the cheaper one and if not, he has a strict preference for the cheaper one.
- In context (iv.), if we are interested in the grades of students -A, B, C and D- there will be 4 scenarios, one scenario for each grade per student. Then, it is natural to assume that $\forall x$, y and $\forall i \in \{1, 2, 3\}, f_{i+1}(x) - f_i(x) = f_{i+1}(y) - f_i(y)$.
- In context (v.), if special types of games is used, for instance lotteries where $f_i(x) = i * f_1(x), \forall x, i$.

Please note that, up to now the interest of using n-point intervals in applications where the data is in form of n numbers, has been underlined. However, for a decision problem, there are two cases in which a preference structure could be needed (see Vincke, 2001):

- comparison: in the case of numerical data, representation illustrates how to compare these values in case of a special preference structure. The examples given up to now, are related to this issue.
- numerical representation: in the case of no numerical data but comparative input. For instance, the decision-makers or experts express their preferences for each pair of objects. The aim is then to associate a numerical representation for each object coherent with the information given by them.

To illustrate the above issue, imagine a decision situation where experts give global judgments (for instance a is indifferent to b which is indifferent to c and c is preferred to a) on certain objects. It could then be interesting to associate numerical values to objects and establish comparison rules. The latter example with a, b and c is a classical one which introduces interval orders since it violates the transitivity of indifference and renders the use of single values impossible. In this case intervals can be associated to each object (for instance [0, 2] to a, [1, 4] to b and [3, 5] to c) and one object is preferred to another if its interval is completely to the right of the other interval (without any intersection). All the remaining situations give place to indifference. Numerous researchers have studied this subject, the majority being interested in the minimal representation of structures (Bogart & Trenk, 1994; Fishburn, 1985; Pirlot & Vincke, 1997). In such cases, coherence conditions reflect a mathematical property and could facilitate the computation of the minimal numerical representation.

Section 2 introduces basic notions of the general framework with n ordered points. Section 3 shows the coherence conditions. Section 4 deals with the results concerning 2-point, 3-point and 4-point intervals under two coherence conditions. Section 5 concludes the article. Section 6 (Annexe) presents definitions of some preference structures (Section 6.1) and proofs (Section 6.2).

2 In all these contexts, if one object has the same evaluation many times, it can be represented by less than n points. The results of the study can be extended in such a case (objects having different number of points) if we keep in mind which points correspond to more than one situation.

3 For instance the one of semiorders implies that $\forall x, y, z, w, xPy, yPz, zIw$ implies xPw.